

ON SUBORDINATION OF HOLOMORPHIC SEMIGROUPS

ALEXANDER GOMILKO AND YURI TOMILOV

ABSTRACT. We prove that for any Bernstein function ψ the operator $-\psi(A)$ generates a holomorphic C_0 -semigroup $(e^{-t\psi(A)})_{t \geq 0}$ on a Banach space, whenever $-A$ does. This answers a question posed by Kishimoto and Robinson. Moreover, giving a positive answer to a question by Berg, Boyadzhiev and de Laubenfels, we show that $(e^{-t\psi(A)})_{t \geq 0}$ is holomorphic in the holomorphy sector of $(e^{-tA})_{t \geq 0}$, and if $(e^{-tA})_{t \geq 0}$ is sectorially bounded in this sector then $(e^{-t\psi(A)})_{t \geq 0}$ has the same property. We also obtain new sufficient conditions on ψ in order that, for every Banach space X , the semigroup $(e^{-t\psi(A)})_{t \geq 0}$ on X is holomorphic whenever $(e^{-tA})_{t \geq 0}$ is a bounded C_0 -semigroup on X . These conditions improve and generalize well-known results by Carasso-Kato and Fujita.

1. INTRODUCTION

The present paper concerns operator-theoretic and function-theoretic properties of Bernstein functions and solves several notable problems which have been left open for some time.

Bernstein functions play a prominent role in probability theory and operator theory. One of their characterizations, also important for our purposes, says that a function $\psi : [0, \infty) \rightarrow (0, \infty)$ is Bernstein if and only if there exists a vaguely continuous semigroup of subprobability measures $(\mu_t)_{t \geq 0}$ on $[0, \infty)$ such that

$$(1.1) \quad e^{-t\psi(z)} = \int_0^\infty e^{-zs} \mu_t(ds), \quad z \geq 0,$$

for all $t \geq 0$.

Let now $(e^{-tA})_{t \geq 0}$ be a C_0 -semigroup on a Banach space X with generator $-A$. The relation (1.1) suggests a way to define a new bounded C_0 -semigroup $(e^{-tB})_{t \geq 0}$ on X in terms of $(e^{-tA})_{t \geq 0}$ and a Bernstein function ψ as

$$(1.2) \quad e^{-tB} = \int_0^\infty e^{-sA} \mu_t(ds),$$

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where $(\mu_t)_{t \geq 0}$ is a semigroup of measures given by (1.1). Following (1.1), it is natural to define $\psi(A) := B$. As it will be revealed in Subsection 4.4 below, such a definition of $\psi(A)$ goes far beyond formal notation and it respects some rules for operator functions called functional calculus.

The semigroup $(e^{-t\psi(A)})_{t \geq 0}$ is subordinated to the semigroup $(e^{-tA})_{t \geq 0}$ via a subordinator $(\mu_t)_{t \geq 0}$. The basics of subordination theory was set up by Bochner [5] and Phillips [41]. This approach to constructing semigroups is motivated by probabilistic applications, e.g. by the study of Lévy processes, but it has also significant value for PDEs as well. As a textbook example one may mention a classical result of Yosida expressing $(e^{-tA^\alpha})_{t \geq 0}$, $\alpha \in (0, 1)$, in terms of $(e^{-tA})_{t \geq 0}$ as in (1.2). The essential feature of this example is that C_0 -semigroups $(e^{-tA^\alpha})_{t \geq 0}$ turn out to be necessarily holomorphic. This fact stimulated further research on relations between functional calculi and Bernstein functions. Some of them are described below.

An easy consequence of (1.2) is that for a fixed Bernstein function ψ the mapping

$$(1.3) \quad \mathcal{M} : -A \mapsto -\psi(A)$$

preserves the class of generators of bounded C_0 -semigroups, and it is natural to ask whether there are any other important classes of semigroup generators stable under \mathcal{M} . In particular, whether \mathcal{M} preserves the class of holomorphic C_0 -semigroups. The question was originally asked by Kishimoto and Robinson in [28, p. 63, Remark]. It appeared to be quite difficult and there have been very few general results in this direction so far.

First, one should probably recall an old and related result due to Hirsch [22] saying that \mathcal{M} maps sectorial operators into sectorial operators if ψ is a *complete* Bernstein function. (In this case, the definition of $\psi(A)$ relies on certain integral representations involving resolvents, see Subsection 4.3). Note that Hirsch's argument does not give any control over the angles of sectoriality.

A partial answer to the Kishimoto-Robinson question was obtained in [4] where the question was formulated in another form: whether \mathcal{M} preserves the class of sectorially bounded holomorphic C_0 -semigroups? It was proved in [4, Theorem 7.2] that for any Bernstein function ψ the operator $-\psi(A)$ generates a sectorially bounded holomorphic C_0 -semigroup of angle $\pi/2$, whenever $-A$ does. Moreover, if $-A$ generates a sectorially bounded holomorphic C_0 -semigroup of angle greater than $\pi/4$ then $-\psi(A)$ is the generator of a sectorially bounded holomorphic C_0 -semigroup as well, [4, Proposition 7.4]. However, in the latter case, the relations between the sectors of holomorphy of the two semigroups was not made precise in [4].

An affirmative answer to the Kishimoto-Robinson question for uniformly convex Banach spaces X was obtained in [33]-[35] using Kato-Pazy's characterization of holomorphic C_0 -semigroups on uniformly convex spaces. In fact, a positive answer to the question in its full generality was also claimed in [34]. However, there seem to be an error in the arguments there (see

Remark 6.7 for more on that), and moreover the permanence of sectors and thus the sectorial boundedness of semigroups has not been addressed in [33]-[35].

Another class of problems related to \mathcal{M} concerns Bernstein functions ψ yielding semigroups $(e^{-t\psi(A)})_{t \geq 0}$ with better properties than the initial semigroup $(e^{-tA})_{t \geq 0}$, as in Yosida's example with $\psi(z) = z^\alpha, \alpha \in (0, 1)$. In particular, it is of value to know when Bernstein functions transform generators of bounded C_0 -semigroups into generators of bounded holomorphic C_0 -semigroups. (Here the boundedness of semigroup is assumed only on the real half-line.) This property of Bernstein functions will further be referred to as the *improving* property. The first results on the improving property are due to Carasso and Kato, [6]. In particular, [6, Theorem 4] gives a criterion for the improving property of ψ in terms of the semigroup $(\mu_t)_{t \geq 0}$ corresponding to ψ and also a necessary condition for that property in terms of ψ itself. Note that while the characterization for the improving property in terms of $(\mu_t)_{t \geq 0}$ exists, it can hardly be applied directly since it is, in general, highly nontrivial to construct $(\mu_t)_{t \geq 0}$ corresponding to ψ . Thus it is desirable to have direct characterizations or conditions for the improving property of ψ .

Certain sufficient conditions for the improving property of ψ were obtained in [13], [43], [34] and [36]. Nontrivial applications of the improving conditions from [6] and [13] can be found in [14], [7] and [29]. We note also [9] where similar results were obtained in a discrete setting.

Our approach to the two problems on \mathcal{M} mentioned above relies on certain extensions of the theory of Bernstein functions and its applications to operator norm estimates by means of functional calculi. Observe that the problems are comparatively simple if ψ is a *complete* Bernstein function, [4]. Thus it is natural to try to use this partial answer in a more general setting of Bernstein functions. Our main idea relies on comparing a fixed Bernstein function ψ to a complete Bernstein function φ associated to ψ in a unique way. It appears that the functions ψ and φ are intimately related and the behavior of ψ and its transforms match in a natural sense the behavior of φ and the corresponding transforms. So our aim is to show that for appropriate λ the "resolvent" functions $(\psi + \lambda)^{-1}$ and $(\varphi + \lambda)^{-1}$ differ by a summand with good integrability (and other analytic) properties and then to recast this fact in terms of functional calculi. The latter step is not however direct and to perform it correctly and transparently we have to use an interplay between several well-known calculi. Apart from answering the questions from [28] and [4], another advantage of our approach is that we have a good control over fine properties of $\psi(A)$, thus deriving the property of permanence of angles under the map \mathcal{M} .

Our functional calculus approach leads, in particular, to the following statement which is one of the main results in this paper.

Theorem 1.1. *Let $-A$ be the generator of a bounded holomorphic C_0 -semigroup of angle $\theta \in (0, \pi/2]$ on a Banach space X . Then for every Bernstein function ψ the operator $-\psi(A)$ generates a bounded holomorphic C_0 -semigroup of angle θ on X as well. Moreover, if $-A$ generates a sectorially bounded holomorphic C_0 -semigroup of angle θ , then the same is true for $-\psi(A)$.*

The functional calculus ideas allow one also to *characterize* the improving property of ψ if ψ is a complete Bernstein function, i.e. if, in addition, ψ extends to the upper half-plane and maps it into itself. The characterization given in Corollary 7.10 below is a consequence of the following interesting result (see Theorem 7.9).

Theorem 1.2. *Let ψ be a complete Bernstein function and let $\gamma \in (0, \pi/2)$ be fixed. The the following assertions are equivalent.*

- (i) *the function ψ maps the closed right half-plane into the sector $\overline{\Sigma}_\gamma := \{\lambda \in \mathbb{C} : |\arg(\lambda)| \leq \gamma\}$.*
- (ii) *For each (complex) Banach space X and each generator $-A$ of a bounded C_0 -semigroup on X , the operator $-\psi(A)$ generates a sectorially bounded holomorphic C_0 -semigroup on X of angle $\pi/2 - \gamma$.*

Moreover, we are able to strengthen essentially the results by Fujita from [13] removing in particular several assumptions made in [13].

Theorem 1.3. *Let ψ be Bernstein function. Suppose there exist $\theta \in (\pi/2, \pi)$ and $r > 0$ such that ψ admits a continuous extension to $\overline{\Sigma}_\theta$ which is holomorphic in Σ_θ , and*

$$(1.4) \quad 0 \leq \arg(\psi(\lambda)) \leq \pi/2 \quad \text{if} \quad 0 \leq \arg(\lambda) \leq \theta \quad \text{and} \quad |\lambda| \geq r.$$

If $-A$ is the generator of a bounded C_0 -semigroup on X , then the (bounded) C_0 -semigroup $(e^{-t\psi(A)})_{t \geq 0}$ is holomorphic in Σ_{θ_0} with $\theta_0 = \frac{\pi}{2}(1 - \pi/(2\theta))$.

Finally, let us describe the structure of our paper. The paper is organized is as follows. Section 3 contains basic information on Bernstein functions together with new notions and properties which are related to Bernstein functions and are crucial for the sequel. In Section 4, we review functional calculi theory needed for the proofs of our main results. Several estimates for resolvents of operators given by complete Bernstein functions of semigroup generators are contained in Section 5. They are probably of some independent interest. Section 6 is devoted to the proof of one of our central results, Theorem 1.1. In Section 7, we study the improving properties of Bernstein functions and complement and strengthen the corresponding statements by Carasso-Kato and Fujita. Finally, in Appendix, we comment on alternative ways to prove Theorem 1.1.

2. NOTATIONS AND GENERALITIES

For a closed linear operator A on a complex Banach space X we denote by $\text{dom}(A)$, $\text{ran}(A)$, $\rho(A)$ and $\sigma(A)$ the *domain*, the *range*, the *resolvent set* and

the *spectrum* of A , respectively, and let $\overline{\text{ran}}(A)$ stand for the norm-closure of the range of A . For closable A we denote its closure by \overline{A} . The space of bounded linear operators on X is denoted by $\mathcal{L}(X)$.

The Laplace transform $\widehat{\mu}$ of a Laplace transformable measure μ will be defined as usual as

$$\widehat{\mu}(z) := \int_0^\infty e^{-sz} \mu(ds)$$

for appropriate z . The same notation and definition will be clearly apply for Laplace transformable functions as well.

For linear operators A and B on X , as usual, we consider the sum $A + B$ and product AB with domains given by

$$\begin{aligned} \text{dom}(A + B) &= \text{dom}(A) \cap \text{dom}(B), \\ \text{dom}(AB) &= \{x \in \text{dom}(B) : Bx \in \text{dom}(A)\}. \end{aligned}$$

We will write the Lebesgue integral $\int_{(0,\infty)}$ as \int_{0+}^∞ . The symbol $*$ will denote convolution of measures (or functions).

Let also

$$\mathbb{C}_+ = \{\lambda \in \mathbb{C} : \text{Re } \lambda > 0\}, \quad \mathbb{R}_+ = [0, \infty),$$

and for $\beta \in (0, \pi]$ and $R > 0$, we denote

$$\Sigma_\beta := \{z \in \mathbb{C} : |\arg \lambda| < \beta\}, \quad \Sigma_0 = (0, \infty),$$

and

$$\Sigma_\beta(R) := \{z \in \Sigma_\beta : |z| < R\}, \quad \Sigma_\beta^+ := \{\lambda \in \mathbb{C} : 0 < \arg \lambda < \beta\}.$$

Finally, we let

$$H^+ := \{\lambda \in \mathbb{C} : \text{Im } \lambda > 0\}.$$

3. BERNSTEIN FUNCTIONS

This section will lay a function-theoretical background for our functional calculi considerations in the subsequent sections. In particular, we will prove a number of new properties of Bernstein functions and revisit some of known ones crucial for the sequel.

We start with recalling one of possible definitions of a Bernstein function.

Definition 3.1. A smooth function $\psi : [0, \infty) \mapsto [0, \infty)$ is called *Bernstein* if its derivative ψ' is *completely monotone*, i.e.

$$(3.1) \quad \psi'(\lambda) = \int_0^\infty e^{-\lambda s} \nu(ds), \quad \lambda > 0,$$

for a Laplace transformable positive Radon measure ν on $[0, \infty)$.

The class of Bernstein functions will be denoted by \mathcal{BF} .

By [45, Theorem 3.2], ψ is Bernstein if and only if there exist $a, b \geq 0$ and a positive Radon measure μ on $(0, \infty)$ satisfying

$$\int_{0+}^\infty \frac{s}{1+s} \mu(ds) < \infty$$

such that

$$(3.2) \quad \psi(z) = a + bz + \int_{0+}^{\infty} (1 - e^{-zs})\mu(ds), \quad z > 0.$$

The formula (3.2) is called the Lévy-Hintchine representation of ψ . The triple (a, b, μ) is defined uniquely and is called Lévy triple of ψ . We will then often write $\psi \sim (a, b, \mu)$ meaning the Lévy-Hintchine representation of ψ . Every Bernstein function extends analytically to \mathbb{C}_+ and continuously to $\overline{\mathbb{C}}_+$. In the following Bernstein functions will be identified with their continuous extensions to $\overline{\mathbb{C}}_+$.

The standard examples of Bernstein functions include

$$(3.3) \quad 1 - e^{-z}, \log(1 + z), \text{ and } z^\alpha, \alpha \in [0, 1].$$

There is a profound theory of Bernstein functions with many implications in functional analysis and probability theory. Being unable to give any reasonable account of them, we refer to a recent book [45].

Geometric properties of Bernstein functions will be of particular importance for us, in particular, the fact that a Bernstein function ψ preserve sectors Σ_ω in a sense that $\psi(\overline{\Sigma}_\omega) \subset \overline{\Sigma}_\omega$, see [45, Proposition 3.6] and [12, Corollary 3.3]. For a later use, we state this result as a proposition below and provide it with a simple proof using an idea from [12].

Proposition 3.2. *Let $F \in \mathcal{BF}$. Then F preserves angular sectors, i.e.*

$$(3.4) \quad F(\overline{\Sigma}_\omega) \subset \overline{\Sigma}_\omega, \quad \omega \in (0, \pi/2).$$

Proof. Note that

$$1 - e^{-z} = \frac{2z}{\pi} \int_0^\infty \frac{(1 - \cos u) du}{u^2 + z^2}, \quad \operatorname{Re} z > 0.$$

Since for every $u > 0$ the function

$$g_u(z) := \frac{z}{u^2 + z^2} = \left(z + \frac{u^2}{z} \right)^{-1}, \quad \operatorname{Re} z > 0,$$

preserves the angular sectors, we infer that the function

$$z \mapsto 1 - e^{-z}, \quad z \in \overline{\mathbb{C}}_+$$

preserves the angular sectors too. Then, using the Levy-Khinchine representation (3.2) of ψ , we obtain (3.4). \square

The preservation of sectors property has further consequences important for the proofs of our main assertions. One of them is mentioned below together with another “geometric” property.

Proposition 3.3. *Let $\psi \in \mathcal{BF}$.*

(i) *For all $\gamma > 0$, $\beta \in (0, \pi/2)$ such that $\gamma + \beta < \pi$,*

$$(3.5) \quad |z + \psi(\lambda)| \geq \cos((\gamma + \beta)/2) (|z| + |\psi(\lambda)|), \quad z \in \overline{\Sigma}_\gamma, \quad \lambda \in \overline{\Sigma}_\beta.$$

(ii) *one has*

$$(3.6) \quad \operatorname{Re} \psi(\lambda) \geq \psi(\operatorname{Re} \lambda), \quad \lambda \in \mathbb{C}_+.$$

Proof. Note first that if $\beta \in (-\pi, \pi)$ and $s > 0$ then

$$(3.7) \quad |1 + se^{i\beta}|^2 = 1 + s^2 + 2s \cos \beta \geq \cos^2 \beta / 2(1 + s)^2.$$

Let now $\gamma > 0$, $\beta > 0$, $\gamma + \beta < \pi$, and

$$z = re^{i\gamma_0} \in \overline{\Sigma}_\gamma, \quad \lambda = \rho e^{i\beta_0} \in \overline{\Sigma}_\beta, \quad |\gamma_0| \leq \gamma, \quad |\beta_0| \leq \beta.$$

Then, using (3.7), we obtain

$$|z + \lambda| = r|1 + r^{-1}\rho e^{i(\beta_0 - \gamma_0)}| \geq \cos((\beta_0 - \gamma_0)/2) (|z| + |\lambda|).$$

From this, since

$$|\beta_0 - \gamma_0| \leq \beta + \gamma \in (0, \pi), \quad \text{and} \quad \cos((\beta_0 - \gamma_0)/2) \geq \cos((\beta + \gamma)/2),$$

it follows that

$$(3.8) \quad |z + \lambda| \geq \cos((\beta + \gamma)/2) (|z| + |\lambda|), \quad z \in \overline{\Sigma}_\gamma, \quad \lambda \in \overline{\Sigma}_\beta.$$

Now (i) is a direct consequence of Proposition 3.2 and (3.8). To prove (ii) it suffices to note that

$$\operatorname{Re} (1 - e^{-\lambda}) \geq 1 - e^{-\operatorname{Re} \lambda}, \quad \lambda \in \mathbb{C}_+,$$

and use the Lévy-Hintchine representation for ψ . □

Recall that every Bernstein function ψ satisfies

$$(3.9) \quad \psi(\lambda) \leq \psi(s\lambda) \leq s\psi(\lambda), \quad \lambda > 0, \quad s \geq 1,$$

and

$$(3.10) \quad \lambda^k \psi^{(k)}(\lambda) \leq k! \psi(\lambda), \quad \lambda > 0, \quad k \in \mathbb{N},$$

see [25, p. 205].

The next lemma provides growth estimates for holomorphic extensions of Bernstein functions to the right half-plane.

Lemma 3.4. *Let $\psi \in \mathcal{BF}$.*

(i) *For all $\beta \in (0, \pi/2)$ and $t > 0$,*

$$(3.11) \quad |\psi(te^{\pm i\beta})| \geq \psi(t \cos \beta).$$

(ii) *There exist $c_\psi > 0$ such that*

$$(3.12) \quad |\psi(z)| \leq c_\psi |z|, \quad z \in \overline{\mathbb{C}}_+, \quad |z| \geq 1.$$

(iii) *For all $\beta \in (0, \pi/2)$*

$$(3.13) \quad |\psi(z)| \geq |z| \psi'(1) \cos \beta, \quad z \in \overline{\Sigma}_\beta, \quad |z| \leq 1.$$

Proof. Let $\psi \sim (a, b, \mu)$. Then

$$\begin{aligned} |\psi(te^{\pm i\beta})| &\geq \operatorname{Re} \psi(te^{i\beta}) \\ &= a + bt \cos \beta + \int_{0+}^{\infty} \operatorname{Re}(1 - e^{-ste^{i\beta}}) \mu(ds) \\ &\geq a + bt \cos \beta + \int_{0+}^{\infty} (1 - e^{-st \cos \beta}) \mu(ds) = \psi(t \cos \beta), \end{aligned}$$

that is (3.11) holds.

To prove (ii), we note that (3.2) yields

$$|\psi(z)| \leq a + b|z| + |z| \int_{0+}^1 s \mu(ds) + 2 \int_1^{\infty} \mu(ds), \quad z \in \overline{\mathbb{C}}_+,$$

and then

$$(3.14) \quad |\psi(z)| \leq c_\psi |z|, \quad z \in \overline{\mathbb{C}}_+, \quad |z| \geq 1.$$

Furthermore, by (i) and (3.9), we have for any $\beta \in (0, \pi/2)$:

$$|\psi(z)| \geq \cos \beta \psi(|z|), \quad z \in \overline{\Sigma}_\beta, \quad |z| \leq 1.$$

Since by (3.10),

$$\psi(z) \geq z\psi'(z) \geq \psi'(1)z, \quad z \in (0, 1],$$

we obtain (3.13). \square

One more notion related to Bernstein functions will also be needed in the sequel. Let us recall (see e.g. [45, Definition 5.24]) that a function $f : (0, \infty) \mapsto (0, \infty)$ is said to be potential, if $f = 1/\psi$, where $\psi \in \mathcal{BF}$. The set of all potentials will be denoted by \mathcal{P} . Note that \mathcal{P} consists precisely of completely monotone functions f satisfying $1/f \in \mathcal{BF}$.

It is often convenient to restrict one's attention to a rich subclass of Bernstein functions formed by complete Bernstein functions. It has a rich structure which makes it especially useful in applications. A Bernstein function ψ is said to be a *complete Bernstein function* if the measure μ in its Lévy-Hintchine representation (3.2) has a completely monotone density m with respect to Lebesgue measure. The set of all complete Bernstein functions will be denoted by \mathcal{CBF} .

The class of complete Bernstein functions allows a number of characterizations. The ones relevant for our purposes are summarized in the following statement, see e.g. [45, Theorem 6.2].

Theorem 3.5. *Let ψ be a non-negative function on $(0, \infty)$. Then the following conditions are equivalent.*

- (i) $\psi \in \mathcal{CBF}$,
- (ii) *There exists a Bernstein function φ such that*

$$(3.15) \quad \psi(\lambda) = \lambda^2 \widehat{\varphi}(\lambda), \quad \lambda > 0.$$

(iii) ψ admits a holomorphic extension to H^+ such that

$$\operatorname{Im}(\psi(\lambda)) \geq 0 \quad \text{for all } \lambda \in H^+,$$

and such that the limit

$$\psi(0+) = \lim_{\lambda \rightarrow 0+} \psi(\lambda)$$

exists.

(iv) ψ admits a holomorphic extension to $\mathbb{C} \setminus (-\infty, 0]$ which is given by

$$(3.16) \quad \psi(\lambda) = a + b\lambda + \int_{0+}^{\infty} \frac{\lambda \sigma(ds)}{\lambda + s},$$

where $a, b \geq 0$ and σ is a positive Radon measure on $(0, \infty)$ such that

$$(3.17) \quad \int_{0+}^{\infty} \frac{\sigma(ds)}{1+s} < \infty.$$

The triple (a, b, σ) is defined uniquely and it is called the Stieltjes representation of ψ .

Using the above result it is easy to see that the first function in (3.3) is not complete Bernstein, while the other Bernstein functions there are clearly complete.

The next statement sharpens Proposition 3.2 in a specific situation when complete Bernstein function has its range in a sector smaller than the right half-plane.

Proposition 3.6. *Let $\psi \in \mathcal{CBF}$ and suppose that*

$$(3.18) \quad \psi(\overline{\mathbb{C}}_+) \subset \overline{\Sigma}_\gamma$$

for some $\gamma \in (0, \pi/2)$. Let $\theta_0 \in (\pi/2, \pi)$ be defined by

$$(3.19) \quad |\cos \theta_0| = \frac{\cot \gamma}{1 + \cot \gamma}.$$

Then for every $\theta \in (\pi/2, \theta_0)$ one has

$$(3.20) \quad \psi(\overline{\Sigma}_\theta) \subset \overline{\Sigma}_{\tilde{\theta}},$$

where

$$\cot \tilde{\theta} = \frac{1 + \cot \gamma}{\sin \theta} \left(\frac{\cot \gamma}{1 + \cot \gamma} - |\cos \theta| \right), \quad \tilde{\theta} \in (0, \pi/2).$$

Proof. By (3.18) it follows that ψ has the Stieltjes representation $(a, 0, \sigma)$. Note that

$$(3.21) \quad \begin{aligned} \psi(re^{i\theta}) &= a + \int_0^\infty \frac{r(r + t \cos \theta) \sigma(dt)}{r^2 + t^2 + 2rt \cos \theta} \\ &\quad + i \sin \theta \int_0^\infty \frac{rt \sigma(dt)}{r^2 + t^2 + 2rt \cos \theta}, \quad r > 0, \quad |\theta| < \pi, \end{aligned}$$

and

$$\operatorname{Im} \psi(re^{i\theta}) > 0, \quad r > 0, \quad \theta \in (0, \pi).$$

Setting in (3.21) the value $\theta = \pi/2$ and using (3.18), we infer that

$$(3.22) \quad a + \int_0^\infty \frac{r^2 \sigma(dt)}{r^2 + t^2} \geq \cot \gamma \int_0^\infty \frac{rt \sigma(dt)}{r^2 + t^2}, \quad r > 0.$$

Moreover, note that for every $\theta \in [\pi/2, \pi)$, and all $r, t > 0$,

$$(3.23) \quad \frac{1}{r^2 + t^2} \leq \frac{1}{r^2 + t^2 + 2rt \cos \theta} \leq \frac{1}{(1 - |\cos \theta|)(r^2 + t^2)}, \quad r, t > 0.$$

Hence, if $\theta \in [\pi/2, \theta_0]$, where θ_0 is given by (3.19), then by (3.21), (3.23) and (3.22) we obtain

$$\begin{aligned} \operatorname{Re} \psi(re^{i\theta}) &\geq a + \int_0^\infty \frac{r^2 - rt|\cos \theta|}{r^2 + t^2 + 2rt \cos \theta} \sigma(dt) \\ &\geq a + \int_0^\infty \frac{r^2 \sigma(dt)}{r^2 + t^2} - \frac{|\cos \theta|}{1 - |\cos \theta|} \int_0^\infty \frac{rt \sigma(dt)}{r^2 + t^2} \\ &\geq \left(\cot \gamma - \frac{|\cos \theta|}{1 - |\cos \theta|} \right) \int_0^\infty \frac{rt}{r^2 + t^2} \sigma(dt) \\ &\geq \left(\cot \gamma - \frac{|\cos \theta|}{1 - |\cos \theta|} \right) (1 - |\cos \theta|) \int_0^\infty \frac{rt \sigma(dt)}{r^2 + t^2 + 2rt \cos \theta} \\ &= \alpha(\theta) \operatorname{Im} \psi(re^{i\theta}), \end{aligned}$$

where

$$\alpha(\theta) = \frac{(1 - |\cos \theta|)}{\sin \theta} \left(\cot \gamma - \frac{|\cos \theta|}{1 - |\cos \theta|} \right) = \frac{1 + \cot \gamma}{\sin \theta} \left(\frac{\cot \gamma}{1 + \cot \gamma} - |\cos \theta| \right).$$

Note that

$$\alpha(\theta_0) = 0, \quad \alpha(\pi/2) = \cot \gamma \quad \text{and} \quad \alpha(\theta) > 0 \quad \text{if} \quad \theta \in [\pi/2, \theta_0).$$

Moreover,

$$\alpha'(\theta) = \frac{\cot \gamma |\cos \theta| - (1 + \cot \gamma)}{\sin^2 \theta} \leq \frac{1}{\sin^2 \theta} < 0, \quad \theta \in [\pi/2, \theta_0],$$

hence $\alpha(\theta)$ is positive and decreasing on $[\pi/2, \theta_0]$. Therefore, for all $\theta' \in (\pi/2, \theta)$ and $\theta \in (\pi/2, \theta_0)$ we have

$$\operatorname{Re} \psi(re^{i\theta'}) \geq \alpha(\theta') \operatorname{Im} \psi(re^{i\theta'}) \geq \alpha(\theta) \operatorname{Im} \psi(re^{i\theta'}).$$

On the other hand, if $\theta' \in (0, \pi/2]$ then, by our assumption,

$$\operatorname{Re} \psi(re^{i\theta}) \geq \cot \gamma \operatorname{Im} \psi(re^{i\theta'}) \geq \alpha(\theta) \operatorname{Im} \psi(re^{i\theta'}).$$

Thus,

$$\psi(\overline{\Sigma}_\theta^{(+)}) \subset \overline{\Sigma}_{\hat{\theta}}^{(+)},$$

and, in view of $\psi(re^{-i\theta}) = \overline{\psi(re^{i\theta})}$, the assertion (3.20) follows. \square

A direct consequence of Theorem 3.5, (iii), is that

$$\psi \in \mathcal{CBF}, \psi \not\equiv 0, \Leftrightarrow \lambda/\psi(\lambda) \in \mathcal{CBF} \Leftrightarrow \lambda\psi(1/\lambda) \in \mathcal{CBF}.$$

Thus, $0 \neq h \in \mathcal{CBF}$ if and only if $\varphi(\lambda) = \lambda h(1/\lambda) \in \mathcal{CBF}$. In view of the latter property and Theorem 3.5, (ii) the following definition is natural.

Definition 3.7. A function $\varphi \in \mathcal{CBF}$ is said to be associated with $\psi \in \mathcal{BF}$ if

$$(3.24) \quad \varphi(\lambda) = \lambda^{-1} \widehat{\psi}(\lambda^{-1}), \quad \lambda > 0.$$

The notion of associated complete Bernstein function will be of primary importance in this paper, and we will first collect its several properties in Lemma 3.9 below. To this aim, the next auxiliary lemma will be useful.

Lemma 3.8. *Define*

$$(3.25) \quad \Delta(\lambda) := \frac{1}{1+\lambda} - e^{-\lambda}, \quad \lambda \in \mathbb{C}_+.$$

Then

$$(3.26) \quad |\Delta(\lambda)| \leq \frac{4|\lambda|^2}{(1+\operatorname{Re} \lambda)^3}, \quad \lambda \in \mathbb{C}_+.$$

Proof. We use the integral representation from [18, p. 3056, Eq. 4.21]):

$$(3.27) \quad \Delta(\lambda) = \lambda^2 \int_0^\infty e^{-\lambda s} G(s) ds, \quad \lambda \in \mathbb{C}_+,$$

where

$$G(s) = \chi(1-s)(s-1+e^{-s}) + \chi(s-1)e^{-s}, \quad s > 0,$$

and χ stands for the characteristic function of $(0, \infty)$. Since

$$s-1+e^{-s} \leq \frac{s^2}{2}, \quad e^{-s} \leq \frac{2}{(s+1)^2}, \quad s > 0,$$

(3.27) implies that

$$\begin{aligned} |\lambda|^{-2} |\Delta(\lambda)| &\leq \int_0^1 e^{-s \operatorname{Re} \lambda} (s-1+e^{-s}) ds + \int_1^\infty e^{-s \operatorname{Re} \lambda} e^{-s} ds \\ &\leq \frac{e}{2} \int_0^1 e^{-s(\operatorname{Re} \lambda + 1)} s^2 ds + \frac{e^{-\operatorname{Re} \lambda - 1}}{\operatorname{Re} \lambda + 1} \\ &\leq \frac{4}{(\operatorname{Re} \lambda + 1)^3}, \quad \lambda \in \mathbb{C}_+. \end{aligned}$$

□

Lemma 3.9. *Let $\varphi \in \mathcal{CBF}$ be associated with $\psi \in \mathcal{BF}$ and let*

$$(3.28) \quad \psi(\lambda) = a + b\lambda + \int_{0+}^\infty (1 - e^{-\lambda s}) \nu(ds), \quad \lambda > 0.$$

Then

a) φ has the representation

$$(3.29) \quad \varphi(\lambda) = a + b\lambda + \int_{0+}^{\infty} \frac{\lambda s \nu(ds)}{1 + \lambda s}, \quad \lambda > 0.$$

b) the inequality

$$(3.30) \quad \operatorname{Re} \psi(\lambda) \geq \varphi(\operatorname{Re} \lambda), \quad \lambda \in \mathbb{C}_+,$$

holds.

c) the estimate

$$(3.31) \quad |\psi(\lambda) - \varphi(\lambda)| \leq 2|\lambda|^2 \varphi''(\operatorname{Re} \lambda), \quad \lambda \in \mathbb{C}_+,$$

holds.

d) ψ is bounded if and only if φ is bounded, and then for any $\beta \in (0, \pi/2)$,

$$\lim_{\lambda \rightarrow \infty, \lambda \in \Sigma_\beta} \psi(\lambda) = \lim_{\lambda \rightarrow \infty, \lambda \in \Sigma_\beta} \varphi(\lambda).$$

Proof. The assertion a) follows directly from (3.28) and (3.24).

To prove b) we note that

$$1 - e^{-\tau} \geq \frac{\tau}{1 + \tau}, \quad \tau > 0.$$

Then, setting $u = \operatorname{Re} \lambda > 0$, by (3.30), (3.28) and (3.29), we obtain

$$\begin{aligned} \operatorname{Re} \psi(\lambda) \geq \psi(u) &= a + bu + \int_{0+}^{\infty} (1 - e^{-us}) \nu(ds) \\ &\geq a + bu + \int_{0+}^{\infty} \frac{us \nu(ds)}{1 + us} \\ &= \varphi(u), \end{aligned}$$

so that (3.30) holds.

Let us now prove c). Observe that

$$(3.32) \quad \psi(\lambda) - \varphi(\lambda) = \int_{0+}^{\infty} \Delta(\lambda s) \nu(ds), \quad \lambda \in \mathbb{C}_+,$$

where Δ is defined by (3.25), and by (3.29),

$$\varphi''(\lambda) = -2 \int_{0+}^{\infty} \frac{s^2 \nu(ds)}{(1 + \lambda s)^3}.$$

Then, using (3.26), it follows that

$$|\psi(\lambda) - \varphi(\lambda)| \leq \int_{0+}^{\infty} |\Delta(\lambda s)| \nu(ds) \leq 4|\lambda|^2 \int_{0+}^{\infty} \frac{s^2 \nu(ds)}{(1 + us)^3} \leq 2|\lambda|^2 \varphi''(u),$$

which is (3.31).

To prove the first statement in d), it suffices to note that boundedness of either ψ or φ is equivalent to boundedness of a measure ν in (3.28) and (3.29) by Fatou's theorem. Finally, since $|\Delta(\lambda)| \leq 2$, $\lambda \in \mathbb{C}_+$, and

$$\lim_{\lambda \rightarrow \infty, \lambda \in \Sigma_\beta} |\Delta(\lambda)| = 0,$$

for any $\beta \in (0, \pi/2)$, (3.32) implies the second assertion in d) by the bounded convergence theorem. \square

One can also give a counterpart of (3.31) with φ'' replaced by φ' as the following corollary of Lemma 3.9, c) shows.

Corollary 3.10. *Let $\varphi \in \mathcal{CBF}$ is associated with $\psi \in \mathcal{BF}$. Then for every $\beta \in (0, \pi/2)$,*

$$(3.33) \quad |\psi(\lambda) - \varphi(\lambda)| \leq \frac{4|\lambda|}{\cos \beta} \varphi'(\operatorname{Re} \lambda), \quad \lambda \in \overline{\Sigma}_\beta \setminus \{0\}.$$

Proof. Note that $\varphi \in \mathcal{CBF}$ implies $s\varphi''(s) \leq 2\varphi'(s)$, $s > 0$. Using (3.31) and observing that

$$|\lambda| \leq \frac{\operatorname{Re} \lambda}{\cos \beta}, \quad \lambda \in \Sigma_\beta, \quad \beta \in (0, \pi/2),$$

we arrive at (3.33). \square

Now we are ready to prove the main result of this section providing an estimate for the “resolvents” of a Bernstein function and the complete Bernstein function associated to it.

Theorem 3.11. *Let ψ be a Bernstein function and let φ be the complete Bernstein function associated with ψ . Let $\omega \in (\pi/2, \pi)$ and $z \in \Sigma_\omega$ be fixed. If*

$$(3.34) \quad r(\lambda; z) := \frac{1}{z + \psi(\lambda)} - \frac{1}{z + \varphi(\lambda)}, \quad \lambda \in \Sigma_{\pi-\omega},$$

then the function $r(\cdot; z)$ is holomorphic in $\Sigma_{\pi-\omega}$ and for every $\beta \in (0, \pi-\omega)$:

$$(3.35) \quad \int_{\partial \Sigma_\beta} |r(\lambda; z)| \frac{|d\lambda|}{|\lambda|} \leq \frac{8}{\cos^2 \beta \cos^2((\omega + \beta)/2) |z|}, \quad z \in \Sigma_\omega.$$

Proof. Note first that $\pi - \omega \in (0, \pi/2)$. Since by Proposition 3.2, the functions ψ and φ preserve sectors, $z + \psi$ and $z + \varphi$ are not zero at each point from $\Sigma_{\pi-\omega}$. As ψ and φ are holomorphic in \mathbb{C}_+ , the holomorphicity of $r(\cdot, z)$ in $\Sigma_{\pi-\omega}$ follows.

Let now $\beta \in (0, \pi - \omega)$ and $0 \neq \lambda \in \overline{\Sigma}_\beta$, $z \in \Sigma_\omega$. If

$$K = \cos((\omega + \beta)/2),$$

then by Proposition 3.3, (i), we have

$$(3.36) \quad K^2 |r(\lambda; z)| \leq \frac{|\varphi(\lambda) - \psi(\lambda)|}{(|z| + |\psi(\lambda)|)(|z| + |\varphi(\lambda)|)}.$$

Let us estimate the numerator and the denominator in the right hand side of (3.36) separately. By (3.33),

$$|\varphi(\lambda) - \psi(\lambda)| \leq \frac{4|\lambda|}{\cos \beta} \varphi'(\operatorname{Re} \lambda),$$

and, moreover, (3.6) and (3.30) yield

$$\begin{aligned} (|z| + |\psi(\lambda)|)(|z| + |\varphi(\lambda)|) &\geq (|z| + \operatorname{Re} \psi(\lambda))(|z| + \operatorname{Re} \varphi(\lambda)) \\ &\geq (|z| + \varphi(\operatorname{Re} \lambda))^2, \quad r > 0. \end{aligned}$$

Thus, if $\lambda = te^{\pm i\beta}$, $t > 0$, then

$$(3.37) \quad K^2 |r(\lambda; z)| \leq \frac{4t}{\cos \beta} \frac{\varphi'(t \cos \beta)}{(|z| + \varphi(t \cos \beta))^2}.$$

Hence,

$$\begin{aligned} K^2 \int_{\partial \Sigma_\beta} |r(\lambda; z)| \frac{|d\lambda|}{|\lambda|} &\leq \frac{8}{\cos \beta} \int_0^\infty \frac{\varphi'(t \cos \beta) dt}{(|z| + \varphi(t \cos \beta))^2} \\ &\leq \frac{8}{\cos^2 \beta |z|}, \end{aligned}$$

and the estimate (3.35) follows. \square

Corollary 3.12. *If $r(\lambda; z)$ is defined as in Theorem 3.11, then for all $z \in \Sigma_\omega$ and $\lambda \in \Sigma_\beta$:*

$$(3.38) \quad r(\lambda; z) = \frac{1}{2\pi i} \int_{\partial \Sigma_\beta} \frac{r(\mu; z) d\mu}{\mu - \lambda},$$

and

$$(3.39) \quad \int_{\partial \Sigma_\beta} \frac{\lambda^k r(\lambda; z)}{(\lambda + 1)^2} d\lambda = 0, \quad k = 0, 1,$$

where the contour $\partial \Sigma_\beta$ is oriented counterclockwise.

Proof. If ψ is unbounded then φ is unbounded as well by Lemma 3.9, d), so using (3.37), (3.6) and (3.10) we obtain that

$$(3.40) \quad |r(\lambda; z)| = o(1) \quad \text{uniformly in } \lambda \in \overline{\Sigma}_\beta, \quad \lambda \rightarrow \infty,$$

for any $z \in \Sigma_\omega$. If ψ is bounded then (3.40) follows directly from (3.36) and Lemma 3.9, d).

Now (3.40), (3.35) and a standard argument based on Cauchy's integral formula yield the representation (3.38).

Finally, (3.39) is a consequence of (3.40) and (3.35). \square

Finally, we mention a property of Bernstein functions which is at the heart of the notion of subordination. To formulate it, recall that a family of positive Radon measures $(\mu_t)_{t \geq 0}$ on $[0, \infty)$ is called a vaguely continuous convolution semigroup of subprobability measures if for all $t, s \geq 0$,

$$(3.41) \quad \mu_t([0, \infty)) \leq 1, \quad \mu_{t+s} = \mu_t * \mu_s, \quad \text{and} \quad \text{vague-} \lim_{t \rightarrow 0+} \mu_t = \delta_0,$$

where δ_0 stands for the Dirac measure at zero. The following classical result due to Bochner can be found e.g. in [45, Theorem 5.2].

Theorem 3.13. *The function $\psi : [0, \infty) \rightarrow (0, \infty)$ is Bernstein if and only if there exists a vaguely continuous convolution semigroup of subprobability measures $(\mu_t)_{t \geq 0}$ on $[0, \infty)$ such that*

$$(3.42) \quad \hat{\mu}_t(z) = \int_0^\infty e^{-zs} \mu_t(ds) = e^{-t\psi(z)}, \quad z \geq 0,$$

for all $t \geq 0$.

Note that $(\mu_t)_{t \geq 0}$ above is defined uniquely.

4. PRELIMINARIES ON FUNCTIONAL CALCULI

The following discussion of functional calculi may seem to be rather long. However, our arguments depend on all of those calculi essentially, and we do not see any other way to make the arguments transparent than to introduce the calculi and to explore the relations between them.

4.1. Abstract functional calculus and its extensions. We start from very abstract considerations. However, such an approach will allow to present several functional calculi below in a unified manner. For its more comprehensive exposition we refer to [20, Section 1].

Let M be a commutative algebra with unit 1, $N \subset M$ be its subalgebra, and X be a complex Banach space. Let $\Phi : N \mapsto \mathcal{L}(X)$ be a homomorphism. Then the triple (M, N, Φ) is called a (primary) functional calculus over X . If the set $\text{Reg}(N) := \{e \in N : \Phi(e) \text{ is injective}\}$ is not empty, then each member of $\text{Reg}(N)$ is called a regulariser. If $f \in M$ and there is $e \in \text{Reg}(N)$ such that also $ef \in N$, then f is called regularisable and e a regulariser for f . If $\text{Reg}(1) \neq \emptyset$, then the functional calculus is called *proper*. Clearly, if the functional calculus is proper then $\mathcal{N} := \{f \in M : f \text{ is regularisable}\}$ is a subalgebra of M containing N , and for $f \in \mathcal{N}$ we then define

$$(4.1) \quad \begin{aligned} \text{dom}(\Phi_e(f)) &:= \{x \in X : (ef)(A)x \in \text{ran } \Phi(e)\} \\ \Phi_e(f) &:= \Phi(e)^{-1} \Phi(ef) \end{aligned}$$

where $e \in \text{Reg}(N)$ is a regulariser for f . Then $\Phi_e(f)$ is a well-defined closed linear operator on X , and the definition (4.1) is independent of the regulariser e . The mapping $\Phi_e : \mathcal{N} \ni f \mapsto \Phi_e(f)$ defined in (4.1) extends Φ , and one usually writes f instead of $\Phi_e(f)$. The triple (M, \mathcal{N}, Φ_e) is called the extended functional calculus (meaning the extension of Φ from N to \mathcal{N} .) We will use the same terminology if \mathcal{N} is replaced by any of its subalgebras containing N .

The extended functional calculus has a number of natural (expected) properties of functional calculi, and we stress here two of them which will be used regularly in the sequel.

Proposition 4.1. *Let (M, \mathcal{N}, Φ_e) be an extended functional calculus over a Banach space X . Then the following assertions hold.*

- (i) If $B \in \mathcal{L}(X)$ commutes with each $\Phi_e(e)$, $e \in \text{Reg}(N)$, then it commutes with each $\Phi_e(f)$, $f \in \mathcal{N}$.
- (ii) Sum rule: given $f, g \in \mathcal{N}$ one has $\Phi_e(f) + \Phi_e(g) \subset \Phi_e(f + g)$, with the equality if $\Phi_e(g) \in \mathcal{L}(X)$;
- (iii) Product rule: given $f, g \in \mathcal{N}$ one has $\Phi_e(f)\Phi_e(g) \subset \Phi_e(fg)$, with the equality if $\Phi_e(g) \in \mathcal{L}(X)$.

From Proposition 4.1, (iii) it follows that if f is regularizable and e is a regularizer, then

$$(4.2) \quad \text{ran}(\Phi_e(e)) \subset \text{dom}(\Phi_e(f)).$$

4.2. Sectorial operators and holomorphic functional calculus. There are several ways to define a function of a sectorial operator. Probably the most well-known approach to that task is provided by the holomorphic functional calculus. This calculus will be relevant for us, and we will set it up below omitting some crucial details and referring to [20, Sections 1-2] for more information.

Definition 4.2. A closed linear operator A on X is called sectorial of angle $\omega \in [0, \pi)$ (in short: $A \in \text{Sect}(\omega)$) if $\sigma(A) \subset \overline{\Sigma}_\omega$ and

$$M(A, \omega_0) := \sup\{\|\lambda(\lambda - A)^{-1}\| : \lambda \notin \Sigma_{\omega_0}\} < \infty$$

for all $\omega < \omega_0 < \pi$. The number

$$\omega_A := \min\{\omega : A \in \text{Sect}(\omega)\}$$

is called the sectoriality angle of A .

Remark 4.3. Note that we do not require A to be densely defined.

The set of sectorial operators on X will be denoted by Sect .

It is well-known that a linear operator A on X is sectorial if and only if $(-\infty, 0) \subset \rho(A)$ and

$$(4.3) \quad M(A) := \sup_{s>0} s\|(s + A)^{-1}\| < \infty,$$

see e.g. [32, Prop. 1.2.1]. The notation $M(A)$ introduced above will be important in the sequel.

The holomorphic functional calculus for sectorial operators is described in details e.g. in [20] and in [30], and we give only a very short account of it here.

For $\omega \in (0, \pi]$, let

$$H_0^\infty(\Sigma_\omega) := \{f \in \mathcal{O}(\Sigma_\omega) : |f(\lambda)| \leq C \min(|\lambda|^s, |\lambda|^{-s}) \text{ for some } C, s > 0\},$$

where $\mathcal{O}(\Sigma_\omega)$ denotes the space of all holomorphic functions on the sector Σ_ω . Let

$$(4.4) \quad \tau(\lambda) := \frac{\lambda}{(1 + \lambda)^2}$$

and

$$\mathcal{B}(\Sigma_\omega) = \{f \in \mathcal{O}(\Sigma_\omega) : |f(\lambda)| \leq C \max(|\lambda|^s, |\lambda|^{-s}) \text{ for some } C, s > 0\}.$$

Let $A \in \text{Sect}(\theta)$ for some angle θ , and let $\theta < \omega < \pi$. Following the abstract scheme described in subsection 4.1, define

$$N = H_0^\infty(\Sigma_\omega), \quad M = \mathcal{O}(\Sigma_\omega).$$

For $f \in H_0^\infty(\Sigma_\omega)$, we define

$$(4.5) \quad \Phi(f) = f(A) := \frac{1}{2\pi i} \int_\Gamma f(\lambda)(\lambda - A)^{-1} d\lambda,$$

where Γ is the downward oriented boundary of a sector Σ_{ω_0} , with $\theta < \omega_0 < \omega$. This definition is independent of ω_0 , and

$$\Phi : H_0^\infty(\Sigma_\omega) \mapsto \mathcal{L}(X), \quad \Phi(f) = f(A),$$

is an algebra homomorphism (such that $\Phi(\tau) = A(1 + A)^{-2}$). The mapping Φ defines the primary holomorphic functional calculus $(O(\Sigma_\omega), H_0(\Sigma_\omega), \Phi)$.

Now let us assume in addition that A is *injective*, so that $\Phi(\tau) = \tau(A)$ is injective too. Then we can define the corresponding extended functional calculus $(O(\Sigma_\omega), \mathcal{B}(\Sigma_\omega), \Phi_e)$ for A . This calculus is called *the extended holomorphic calculus* for A . Any function $f \in \mathcal{B}(\Sigma_\omega)$ has a regulariser e of the form $e = \tau^n$, thus

$$(4.6) \quad f(A) = [\tau^n(A)]^{-1}(\tau^n f)(A),$$

where $n \in \mathbb{N}$ is so large that

$$\tau^n f \in H_0^\infty(\Sigma_\omega).$$

This functional calculus formally depends on a choice of ω , but the calculi are consistent for different ω 's if we identify a function f on Σ_ω with its restriction to Σ_γ for $\theta < \gamma < \omega$. We may therefore make this identification and consider our holomorphic calculus to be defined on the algebra

$$\mathcal{B}[\Sigma_\omega] := \bigcup_{\omega < \gamma < \pi} \mathcal{B}(\Sigma_\gamma).$$

Note that Proposition 4.1 is clearly holds for the triple $(\mathcal{B}[\Sigma_\omega], O(\Sigma_\omega), \Phi_e)$ and we have in particular the sum rule and the product rule. These rules will often be used without a specific reference.

The assumption of injectivity of A is often rather restrictive. To avoid it, we may consider another subalgebra of $O(\Sigma_\omega)$ defined as

$$\mathcal{B}_0(\Sigma_\omega) := \{f \in \mathcal{O}(\Sigma_\omega) : |f(\lambda)| \leq C|\lambda|^s \text{ for some } C, s > 0\}.$$

Every function $f \in \mathcal{B}_0(\Sigma_\omega)$ has a regulariser e of the form $e = \tau_0^n$ where $\tau_0(z) = 1/(1+z)$ and $n \in \mathbb{N}$ is large enough. Thus using (4.6) as above, we can define the extended functional calculus $(O(\Sigma_\omega), \mathcal{B}_0[\Sigma_\omega], \Phi_e)$ for arbitrary sectorial A . In this way, all fractional powers $A^q, q > 0$, are well-defined, and this definition will be basic for us in dealing with fractional powers of sectorial operators.

We will frequently use the following composition rule for the holomorphic functional calculus, see e.g. [20, Theorem 3.1.4].

Proposition 4.4. *Let $\alpha \in (0, 1)$ and*

$$0 \leq \omega < \beta \leq \pi, \quad \beta' = \alpha\beta < \pi.$$

Suppose that

$$f \in \mathcal{B}(\Sigma_{\beta'}).$$

Let $A \in \text{Sect}(\omega)$ and let A be injective. Then

$$f(A^\alpha) = (f \circ \lambda^\alpha)(A).$$

Finally, we will also need the property of sectoriality of fractional powers of sectorial operators, see e.g. [20, Theorem 3.1.2] for its proof.

Proposition 4.5. *Suppose that $A \in \text{Sect}(\omega)$, $q > 0$, and $q\omega < \pi$. Then $A^q \in \text{Sect}(q\omega)$.*

4.3. Hirsch functional calculus. The definition of the holomorphic functional calculus for a sectorial operator by means of regularisers is quite implicit. On the other hand, it is often useful to have a comparatively simple expression for a function of a sectorial operator allowing for estimates in resolvent terms. To this aim, one may use the Hirsch functional calculus which provides a definition of complete Bernstein function for a sectorial operator.

We now define complete Bernstein functions of sectorial operators following Hirsch and review some of their basic properties needed in the sequel. Let A be a sectorial operator on X . The next definition was essentially given in [22, p. 255], see also [3].

Definition 4.6. Given $f \in \mathcal{CBF}$ with Stieltjes representation (a, b, σ) (see (3.16)), define an operator $f_0(A) : \text{dom}(A) \rightarrow X$ by

$$(4.7) \quad f_0(A)x = ax + bAx + \int_{0+}^{\infty} A(\lambda + A)^{-1}x \, d\sigma(\lambda), \quad x \in \text{dom}(A).$$

By (3.17), this integral is absolutely convergent and $f_0(A)(I + A)^{-1}$ is a bounded operator on X , extending $(I + A)^{-1}f_0(A)$. Hence $f_0(A)$ is closable as an operator on X . Define

$$(4.8) \quad f(A) = \overline{f_0(A)}.$$

We call $f(A)$ a *complete Bernstein function of A* .

Note that by Definition 4.6, $\text{dom}(A)$ is core for $f(A)$.

The mapping $f \mapsto f(A)$ defined by (4.7) possesses a number of properties of functional calculus, see e.g. [22, Théorème 1-3] and [24, Théorème 1] (see also [31, Th. 2.3], [24, p. 200-201] and [23]). Thus we call this mapping the *Hirsch functional calculus* and describe some of its properties below. Note that the product rule holds for the Hirsch functional calculus restrictedly since the set of complete Bernstein functions is not closed under multiplication.

Theorem 4.7. *Let A be a sectorial operator on X , and let f and g be complete Bernstein functions. Then the following statements hold.*

(i) *The operator $f(A)$ (and $g(A)$) is sectorial and*

$$(4.9) \quad \sup_{s>0} \|s(f(A) + s)^{-1}\| \leq \sup_{s>0} \|s(A + s)^{-1}\|.$$

(ii) *The composition rule holds: $f(g(A)) = (f \circ g)(A)$. In particular, if $f_\alpha(\lambda) = f(\lambda^\alpha)$ for some $\alpha \in (0, 1)$, then*

$$(4.10) \quad f(A^\alpha) = f_\alpha(A),$$

and if $g_\beta(\lambda) := [g(\lambda)]^\beta \in \mathcal{CBF}$ for some $\beta > 1$, then

$$(4.11) \quad [g_\beta(A)]^{1/\beta} = g(A).$$

(iii) *If the product fg is also a complete Bernstein function, then $f(A)g(A) = (fg)(A)$.*

If $\alpha \in (0, 1)$, then remark after Theorem 3.5 show that z^α is a complete Bernstein function, and we write A^α for the corresponding complete Bernstein function of A . These fractional powers coincide with the standard fractional powers defined in the framework of the holomorphic functional calculus in the previous subsection, see e.g. [3].

4.4. Hille-Phillips functional calculus. In the functional calculi described in Section 4.2 function of a sectorial operator had to be analytic on the spectrum of the operator. On the other hand, in some cases and in the present considerations as well it is possible to drop this assumption to some extent. In particular, if $-A$ generates a bounded C_0 -semigroup then $A \in \text{Sect}(\pi/2)$, where $\pi/2$ cannot be replaced by a smaller angle, in general. On the other hand, if $f \in L^1(\mathbb{R}_+)$ then its Laplace transform \hat{f} is holomorphic only in $\Sigma_{\pi/2}$, where $\pi/2$ cannot, in general, be replaced by a larger angle. Nevertheless, it is still possible to define $\hat{f}(A)$ by the Hille-Phillips functional calculus described below.

Let us recall definition and basic properties of the (extended) Hille-Phillips functional calculus useful for the sequel.

Let $M_b(\mathbb{R}_+)$ be a Banach algebra of bounded Radon measures on \mathbb{R}_+ . If

$$A_+^1(\mathbb{C}_+) := \{\hat{\mu} : \mu \in M_b(\mathbb{R}_+)\}$$

then $A_+^1(\mathbb{C}_+)$ is a commutative Banach algebra with pointwise multiplication and with the norm

$$(4.12) \quad \|\hat{\mu}\|_{A_+^1(\mathbb{C}_+)} := \|\mu\|_{M_b(\mathbb{R}_+)} = |\mu|(\mathbb{R}_+),$$

where $|\mu|(\mathbb{R}_+)$ stands for the total variation of μ on \mathbb{R}_+ .

Let $-A$ be the generator of a bounded C_0 -semigroup $(e^{-tA})_{t \geq 0}$ on X . Then the mapping

$$A_+^1(\mathbb{C}_+) \mapsto \mathcal{L}(X), \quad \Phi(\hat{\mu})x := \int_0^\infty e^{-sA} x \mu(ds), \quad x \in X,$$

defines a continuous algebra homomorphism Φ such that

$$(4.13) \quad \|\Phi(\hat{\mu})\| \leq \sup_{t \geq 0} \|e^{-tA}\| \|\mu\|(\mathbb{R}_+).$$

The homomorphism Φ is called the (primary) *Hille-Phillips* (HP-) functional calculus for A , and if $g = \hat{\mu}$ then we set

$$g(A) = \Phi(\hat{\mu}).$$

Basic properties of the Hille-Phillips functional calculus can be found in [21, Chapter XV].

As in the case of holomorphic functional calculus, we can define the extended *HP*-calculus by the extension procedure described in Subsection 4.1.

Let $O(\mathbb{C}_+)$ be an algebra of functions holomorphic in \mathbb{C}_+ . If $f \in O(\mathbb{C}_+)$ is holomorphic such that there exists $e \in A_+^1(\mathbb{C}_+)$ with $ef \in A_+^1(\mathbb{C}_+)$ and the operator $e(A)$ is injective, then one defines $f(A)$ as in (4.1). We then call the triple $(O(\mathbb{C}_+), A_+^1(\mathbb{C}_+), \Phi_e)$ the *extended Hille-Phillips calculus* for A .

Using the extended Hille-Phillips functional calculus, one can define a Bernstein function $\psi(A)$ of A and obtain a useful, Lévy-Hintchine type representation for $\psi(A)$. Recall first that Bernstein functions are regularisable by $e(z) = 1/(1+z)$.

Proposition 4.8. [16, Lemma 2.5] *Let $\psi \in \mathcal{BF}$. Then $\psi(z)/(1+z) \in A_+^1(\mathbb{C}_+)$.*

Proposition 4.8 implies that for $\psi \in \mathcal{BF}$ the operator $\psi(A)$ can be defined by the extended *HP*-calculus as

$$(4.14) \quad \psi(A) = (1+A)[\psi(z)(1+z)^{-1}](A)$$

with the natural domain

$$\text{dom}(\psi(A)) = \{x \in X : [\psi(z)(1+z)^{-1}](A)x \in \text{dom}(A)\}.$$

As it was proved in [16, Corollary 2.6], the formula (4.14) can be written in a more explicit and useful form called the Lévy-Hintchine representation. Moreover, Bernstein functions of semigroup generators possess certain permanence properties. Namely, for any $\psi \in \mathcal{BF}$ the operator $-\psi(A)$ generates a bounded C_0 -semigroup on X as well and the latter semigroup can be represented in terms of ψ and $(e^{-tA})_{t \geq 0}$ as (3.42) suggests, see e.g. [45, Proposition 13.1 and Theorem 13.6] and [41].

For later reference, we summarize these results below.

Theorem 4.9. (i) *Let $-A$ generate a bounded C_0 -semigroup $(e^{-tA})_{t \geq 0}$ on X , and let ψ be a Bernstein function with the corresponding Lévy-Hintchine representation (a, b, μ) . Then $\psi(A)|_{\text{dom}(A)}$ is given by*

$$(4.15) \quad \psi(A)x = ax + bAx + \int_{0+}^{\infty} (1 - e^{-sA})x \mu(ds), \quad x \in \text{dom}(A),$$

where the integral is understood as a Bochner integral, and $\text{dom}(A)$ is core for $\psi(A)$.

- (ii) Moreover, $-\psi(A)$ is the generator of a bounded C_0 -semigroup $(e^{-t\psi(A)})_{t \geq 0}$ on X given by

$$e^{-t\psi(A)} := \int_0^\infty e^{-sA} \mu_t(ds), \quad t \geq 0,$$

where $(\mu_t)_{t \geq 0}$ is a vaguely continuous convolution semigroup of sub-probability measures on $[0, \infty)$ corresponding to ψ by (3.41) (cf. (3.42)).

The semigroup $(e^{-t\psi(A)})_{t \geq 0}$ defined in Theorem 4.9 is called *subordinate* to the semigroup $(e^{-tA})_{t \geq 0}$ with respect to the Bernstein function ψ . Theorem 4.9, (ii), implies that $(-\infty, 0) \subset \rho(\psi(A))$ and

$$(4.16) \quad \sup_{s > 0} \|s(s + \psi(A))^{-1}\| \leq \sup_{t > 0} \|e^{-t\psi(A)}\| \leq \sup_{t > 0} \|e^{-tA}\|.$$

The next approximation result will often allow us to reduce considerations to the case when the semigroup generator is invertible.

Proposition 4.10. *Let A be the generator of a bounded C_0 -semigroup on X and let ψ be a Bernstein function.*

- (i) *If $\epsilon > 0$ then $[\psi(\cdot + \epsilon) - \psi(\cdot)](A) \in \mathcal{L}(X)$ so that $\text{dom}(\psi(A + \epsilon)) = \text{dom}(\psi(A))$ and*

$$(4.17) \quad \psi(A + \epsilon) = \psi(A) + [\psi(\cdot + \epsilon) - \psi(\cdot)](A).$$

- (ii) *If $x \in \text{dom}(A)$ and $\epsilon > 0$, then*

$$(4.18) \quad \|\psi(A + \epsilon)x - \psi(A)x\| \leq M(\psi(\epsilon) - \psi(0))\|x\|.$$

where $M := \sup_{t \geq 0} \|e^{-tA}\| \geq 1$.

- (iii) *For every $s > 0$,*

$$(4.19) \quad \lim_{\epsilon \rightarrow 0+} \|(s + \psi(A + \epsilon))^{-1}x - (s + \psi(A))^{-1}x\| = 0, \quad x \in X.$$

Proof. To prove (i) and (ii), we note that if ψ is a Bernstein function with the Lévy-Hintchine representation (a, b, μ) then for all $\epsilon > 0$ and $\lambda \in \mathbb{C}_+$,

$$\psi(\lambda + \epsilon) - \psi(\lambda) = b\epsilon + \int_{0+}^\infty e^{-\lambda s}(1 - e^{-\epsilon s})\mu(ds).$$

Hence, $\psi(\cdot + \epsilon) - \psi(\cdot) \in A_+^1(\mathbb{C}_+)$ and

$$\|\psi(\cdot + \epsilon) - \psi(\cdot)\|_{A_+^1} = \psi(\epsilon) - \psi(0).$$

From here by the (extended) HP-calculus it follows that

$$\|[\psi(\cdot + \epsilon) - \psi(\cdot)](A)\| \leq M(\psi(\epsilon) - \psi(0)).$$

Moreover, since $[\psi(\cdot + \epsilon) - \psi(\cdot)](A) \in \mathcal{L}(X)$, by the sum rule for the (extended) HP-calculus we obtain that $\text{dom}(\psi(A + \epsilon)) = \text{dom}(\psi(A))$ and (4.17) holds. Since $\text{dom}(A) \subset \text{dom}(\psi(A))$ by Theorem 4.9, (i), we have also

$$(4.20) \quad \|\psi(A + \epsilon)x - \psi(A)x\| \leq M[\psi(\epsilon) - \psi(0)]\|x\|,$$

for all $x \in \text{dom}(A)$. This finishes the proof of (i) and (ii).

Furthermore, by the product rule for the (extended) HP-calculus, for all $\epsilon, s > 0$ and $x \in \text{dom}(A)$,

$$(4.21) \quad (s + \psi(A))^{-1}(\psi(A + \epsilon) - \psi(A))x = (\psi(A + \epsilon) - \psi(A))(s + \psi(A))^{-1}x.$$

Hence the estimates (4.16) and (4.20) yield

$$(4.22) \quad \begin{aligned} & \| (s + \psi(A))^{-1}x - (s + \psi(A + \epsilon))^{-1}x \| \\ &= \| (s + \psi(A))^{-1}(s + \psi(A + \epsilon))^{-1}(\psi(A + \epsilon) - \psi(A))x \| \\ &\leq \frac{M^2}{s^2} \| \psi(A + \epsilon)x - \psi(A)x \| \\ &\leq \frac{M^3}{s^2} (\psi(\epsilon) - \psi(0)) \| x \| \\ &\rightarrow 0, \quad \epsilon \rightarrow 0+. \end{aligned}$$

As $\text{dom}(A)$ is dense in X , (iii) follows. \square

Let $\psi \in \mathcal{BF}$. Then ψ is holomorphic in \mathbb{C}_+ and continuous in $\overline{\mathbb{C}}_+$, and by Lemma 3.4, (ii) we have

$$(4.23) \quad \tau^2 \psi \in H_0^\infty(\mathbb{C}_+) \quad \text{and} \quad \tau^2/\psi \in H_0^\infty(\Sigma_\omega), \quad \omega \in (0, \pi/2),$$

where τ is defined by (4.4).

Let $A \in \text{Sect}(\alpha)$ for some $\alpha \in [0, \pi/2)$, so that, in particular, $-A$ generates a sectorially bounded holomorphic C_0 -semigroup. Suppose that A has dense range. Let $f \in \mathcal{P}$, that is $f := 1/\psi$ for some $\psi \in \mathcal{BF}, \psi \neq 0$. Then, by (4.23), the operators $\psi(A)$ and $f(A)$ can be defined by the (extended) holomorphic calculus with the regulariser τ_ϵ of the form

$$(4.24) \quad \tau_\epsilon(z) := \left(\frac{z}{(\epsilon + z)(1 + \epsilon z)} \right)^2,$$

for any fixed $\epsilon > 0$. Thus by [20, Proposition 1.2.2, d], we have

$$(4.25) \quad f(A) = [\psi(A)]^{-1}.$$

Moreover, for every h of the form

$$(4.26) \quad h = \psi + f, \quad \psi \in \mathcal{BF}, \quad f \in \mathcal{P},$$

and every $A \in \text{Sect}(\alpha)$, $\alpha \in [0, \pi/2)$, with dense range, the (closed) operator $h(A)$ is well-defined by the extended holomorphic functional calculus (with the regulariser τ_ϵ , $\epsilon > 0$.)

Proposition 4.11. *Let h be of the form (4.26) and $A \in \text{Sect}(\alpha)$, $\alpha \in [0, \pi/2)$. If A has dense range then*

$$(4.27) \quad \overline{\psi(A) + f(A)} = h(A).$$

Proof. We have

$$\lim_{\epsilon \rightarrow 0} \tau_\epsilon(A)x = x, \quad x \in X,$$

and then the statement follows from [19, Proposition 3.3] or [8, Proposition 3.2]. \square

4.5. Compatibility of functional calculi. This subsection allows us to unify the results of previous subsections and to show that different definitions of the same function of semigroup generator are compatible.

The first result proved in [3, Theorem 4.12] shows that the extended holomorphic functional calculus and the Hirsch functional calculus are compatible.

Proposition 4.12. *Let $\psi \in \mathcal{CBF}$ and let A be an injective sectorial operator on X . Then the operator $\psi(A)$ defined by the Hirsch calculus coincides with $\psi(A)$ defined via the extended holomorphic functional calculus.*

The second result yields compatibility of the extended Hille-Phillips calculus and the extended holomorphic calculus on appropriate functions.

Proposition 4.13. *Let $\psi \in \mathcal{BF}$. Suppose that ψ admits holomorphic extension $\tilde{\psi}$ to Σ_ω for some $\omega \in (\pi/2, \pi)$ so that*

$$\tilde{\psi} \in \mathcal{B}(\Sigma_\omega).$$

Let $-A$ be the generator of a bounded C_0 -semigroup and let A be injective. Let $\psi(A)$ be defined by the extended Hille-Phillips calculus and $\tilde{\psi}(A)$ be defined via the extended holomorphic functional calculus. Then

$$(4.28) \quad \tilde{\psi}(A) = \psi(A).$$

Proof. Note that for some $n \in \mathbb{N}$

$$\tilde{\tau}(\lambda) = \tau^n(\lambda) = \left(\frac{\lambda}{(\lambda+1)^2} \right)^n,$$

is a regulariser for $\tilde{\psi}$ in the holomorphic functional calculus. By Proposition 4.8, the function $\tilde{\tau}$ is a regulariser for ψ in the extended HP -calculus, i.e. there exists a bounded Radon measure μ on \mathbb{R}_+ such that

$$\tilde{\tau} \in A_+^1(\mathbb{C}_+), \quad (\tilde{\tau}\psi)(\lambda) = \int_0^\infty e^{-\lambda s} \mu(ds) \in A_+^1(\mathbb{C}_+).$$

Then, by [20, Proposition 3.3.2] on compatibility of the Hille-Phillips and the holomorphic functional calculi, we have

$$(4.29) \quad (\tilde{\tau}\tilde{\psi})(A) = \int_0^\infty e^{-sA} \mu(ds) = (\tilde{\tau}\psi)(A).$$

By the same [20, Proposition 3.3.2], the operators $\tilde{\tau}(A)$ given by the holomorphic calculus and by the HP -calculus are the same. So, (4.29) implies (4.28). \square

5. RESOLVENT ESTIMATES FOR CERTAIN FUNCTIONS OF SECTORIAL OPERATORS

The following estimates of independent interest will be instrumental in subsequent sections. In a qualitative form, they are essentially known. However, for our purposes, it is crucial to equip them with explicit constants and specify large enough sectors where such estimates hold.

The first estimate is a version of the well-known result on sectoriality of fractional powers due to Kato, [26, Theorem 2]. However we give an explicit constant in the sectoriality condition restricted to an appropriate sector, and this could be helpful in many instances.

Proposition 5.1. *Let $A \in \text{Sect}$, so that*

$$\|s(A + s)^{-1}\| \leq M, \quad s > 0.$$

If $r \in (0, 1)$ and $\gamma \in (0, (1 - r)\pi)$, then

$$(5.1) \quad \|(A^r + z)^{-1}\| \leq \frac{M \sin(\pi r)}{\pi r} \frac{(\pi r + \gamma)}{\sin(\pi r + \gamma)} \frac{1}{|z|}, \quad z \in \Sigma_\gamma.$$

Proof. If $z \in \Sigma_\gamma$, then by [26, Theorem 2] we have

$$(5.2) \quad \begin{aligned} (A^r + z)^{-1} &= \frac{\sin(\pi r)}{\pi} \int_0^\infty \frac{t^r (A + t)^{-1} dt}{(t^r e^{i\pi r} + z)(t^r e^{-i\pi r} + z)} \\ &= \frac{\sin(\pi r)}{\pi} \int_0^\infty \frac{t^r (A + t)^{-1} dt}{t^{2r} + 2t^r z \cos \pi r + z^2}. \end{aligned}$$

Since

$$|t^r e^{\pm i\pi r} + z| \geq |t^r e^{i(\pi r + \gamma)} + |z||, \quad t > 0, \quad z \in \Sigma_\gamma,$$

and

$$\int_0^\infty \frac{dt}{|te^{i\beta} + s|^2} = \frac{\beta}{s \sin \beta}, \quad \beta \in (-\pi, \pi), \quad s > 0$$

(see e.g. ([40, Formula 2.2.9.25]), (5.2) implies that

$$\begin{aligned} \|(A^r + z)^{-1}\| &\leq \frac{M \sin(\pi r)}{\pi} \int_0^\infty \frac{t^{r-1} dt}{|t^r e^{i\pi r} + z| |t^r e^{-i\pi r} + z|} \\ &\leq \frac{M \sin(\pi r)}{\pi} \int_0^\infty \frac{t^{r-1} dt}{|t^r e^{i(\pi r + \gamma)} + |z||^2} \\ &= \frac{M \sin(\pi r)}{\pi r} \int_0^\infty \frac{dt}{|te^{i(\pi r + \gamma)} + |z||^2} \\ &= \frac{M \sin(\pi r)}{\pi r} \frac{\pi r + \gamma}{\sin(\pi r + \gamma)} \frac{1}{|z|}, \quad z \in \Sigma_\gamma. \end{aligned}$$

□

The next result is an explicit version of Proposition 4.5 for $q > 1$. We are not aware of an estimate of this kind in the literature.

Proposition 5.2. *Let operator $A \in \text{Sect}(\alpha)$, $\alpha \in (0, \pi)$, so that*

$$(5.3) \quad \|z(A+z)^{-1}\| \leq M(A, \alpha'), \quad z \in \Sigma_{\pi-\alpha'}, \quad \alpha' \in (\alpha, \pi),$$

and, in particular,

$$(5.4) \quad \|s(A+s)^{-1}\| \leq M(A), \quad s > 0.$$

Let $q > 1$ be such that $q\alpha < \pi$. Then $A^q \in \text{Sect}(q\alpha)$ and, moreover, for every $\gamma \in (0, \pi - q\alpha)$,

$$(5.5) \quad \|z(A^q + z)^{-1}\| \leq \tilde{M}_{\alpha, q; \gamma}, \quad z \in \Sigma_\gamma,$$

where

$$(5.6) \quad \tilde{M}_{\alpha, q; \gamma} := M(A) + \frac{2M(A, \beta_{\alpha, q; \gamma})}{\pi \cos(\beta_{\alpha, q; \gamma}/2) \cos(q\beta_{\alpha, q; \gamma}/2)}, \quad \beta_{\alpha, q; \gamma} := \frac{\alpha + (\pi - \gamma)/q}{2}.$$

Proof. Let $\gamma \in (0, \pi - q\alpha)$ be fixed. Since $f_{s, q} \in H_0^\infty(\Sigma_\pi)$, using the holomorphic functional calculus and [20, Proposition 3.1.2], we have

$$z(A^q + z)^{-1} = |z|^{1/q}(|z|^{1/q} + A)^{-1} + f_{z, q}(A), \quad z \in \Sigma_{\pi - q\alpha},$$

where

$$f_{z, q}(\lambda) := \frac{z}{z + \lambda^q} - \frac{|z|^{1/q}}{\lambda + |z|^{1/q}} = \frac{z\lambda - z^{1/q}\lambda^q}{(z + \lambda^q)(\lambda + |z|^{1/q})}.$$

Hence,

$$\|z(A^q + z)^{-1}\| \leq M(A) + \|f_{z, q}(A)\|, \quad z \in \Sigma_{\pi - q\alpha},$$

Furthermore, if $\beta \in (\alpha, \pi/q)$ and $z \in \Sigma_\gamma$, then

$$\begin{aligned} \|f_{z, q}(A)\| &\leq \frac{1}{2\pi} \int_{\partial S_\beta} |f_{z, q}(\lambda)| \|(\lambda - A)^{-1}\| |d\lambda| \\ &\leq \frac{M(A, \beta)}{2\pi} \int_{\partial S_\beta} |f_{z, q}(\lambda)| \frac{|d\lambda|}{|\lambda|}, \quad \beta \in (\alpha, \pi/q). \end{aligned}$$

Moreover, if, in addition, $q \in (1, \pi/\alpha)$ and $z \in \Sigma_\gamma$, then setting $C = \cos(\beta/2) \cos((q\beta + \gamma)/2)$ and using (3.8), we have

$$\begin{aligned} \int_{\partial S_\beta} |f_{z, q}(\lambda)| \frac{|d\lambda|}{|\lambda|} &\leq \int_{\partial S_\beta} \frac{|z| + |z|^{1/q} |\lambda|^{q-1}}{|z + \lambda^q| |\lambda + |z|^{1/q}|} |d\lambda| \\ &\leq C \int_{\partial S_\beta} \frac{|z| + |z|^{1/q} |\lambda|^{q-1}}{(|z| + |\lambda|^q)(|\lambda| + |z|^{1/q})} |d\lambda| \\ &= 2C \int_0^\infty \frac{1 + t^{q-1}}{(t^q + 1)(t + 1)} dt. \end{aligned}$$

Since $q \geq 1$,

$$\int_0^\infty \frac{(1 + t^{q-1}) dt}{(t^q + 1)(t + 1)} \leq 2 \int_0^\infty \frac{dt}{(t + 1)^2} = 2,$$

thus

$$\|z(A^q + z)^{-1}\| \leq M(A) + \frac{2M(A, \beta)}{\pi \cos(\beta/2) \cos((q\beta + \gamma)/2)}, \quad z \in \Sigma_\gamma.$$

Putting $\beta = \beta_{\alpha, q; \gamma}$, we obtain (5.5) and (5.6). \square

In [4, Theorem 6.1 and Remark 6.2] it was proved that if $\psi \in \mathcal{CBF}$ then

$$(5.7) \quad A \in \text{Sect}(\theta) \implies \psi(A) \in \text{Sect}(\theta), \quad \theta \in [0, \pi/2).$$

The proof of (5.7) there was based on the fact that

$$(5.8) \quad \psi \in \mathcal{CBF} \implies [\psi(\lambda^\alpha)]^{1/\alpha} \in \mathcal{CBF}, \quad \alpha \in (0, 1),$$

and on Theorem 4.7. (See also [38, Corollary 2] and [45, Corollary 7.15].) We present below a slight generalization of (5.7) extending it to the whole class of sectorial operators. Once again, apart from proving the sectoriality of $\psi(A)$, we give explicit constants in the resolvent bounds for $\psi(A)$ and get a control over the sectoriality angle of $\psi(A)$. This will be used in subsequent sections.

Theorem 5.3. *Let $A \in \text{Sect}(\alpha)$, $\alpha \in (0, \pi)$. If $\psi \in \mathcal{CBF}$, then $\psi(A) \in \text{Sect}(\alpha)$ too. Moreover, if $q \in (1, \pi/\alpha)$ and $\gamma \in (0, (1 - q^{-1})\pi)$, then*

$$(5.9) \quad \|z(\psi(A) + z)^{-1}\| \leq \frac{\tilde{M}_{\alpha, q; \gamma}(A) \sin(\pi/q)}{\pi/q} \frac{\pi/q + \gamma}{\sin(\pi/q + \gamma)}, \quad z \in \Sigma_\gamma,$$

where $\tilde{M}_{\alpha, q; \gamma}(A)$ is given by (5.6).

Proof. In this proof, we will combine the (extended) holomorphic functional calculus and the Hirsch functional calculus. This is possible due to compatibility of the calculi given by Proposition 4.12.

Recall first that by Theorem 4.7, (i), the operator $\psi(A)$ is sectorial. Choose $q \in (1, \pi/\alpha)$ so that $1/q \in (0, 1)$, and define $g_{1/q}(z) := z^{1/q}$ and $\eta_q(z) := z^q$. Since $g_q \circ \psi \circ g_{1/q} \in \mathcal{CBF}$ (see (5.8)) and A^q is sectorial in view of Proposition 4.5, we infer by (4.11) that

$$[(g_q \circ \psi \circ g_{1/q})(A^q)]^{1/q} = [\psi \circ g_{1/q}](A^q),$$

and, moreover, by (4.10),

$$[\psi \circ g_{1/q}](A^q) = \psi((A^q)^{1/q}) = \psi(A).$$

Hence

$$(5.10) \quad \varphi := g_q \circ \psi \circ g_{1/q} \in \mathcal{CBF} \quad \text{and} \quad \psi(A) = [\varphi(A^q)]^{1/q},$$

and, by Proposition 4.5, we obtain that

$$(5.11) \quad \psi(A) \in \text{Sect}(\pi/q).$$

As $q \in (1, \pi/\alpha)$ is arbitrary, choosing q closely enough to π/α we can make γ arbitrarily close to $\pi - \alpha$. Thus, from (5.11) it follows that $\psi(A) \in \text{Sect}(\alpha)$.

Let now $\gamma \in (0, (1 - q^{-1})\pi)$ and $z \in \Sigma_\gamma$. Using (5.10), Proposition 5.1 with $r = 1/q$, Theorem 4.7, (i), and Proposition 5.2, we conclude that

$$\begin{aligned} \|z(\psi(A) + z)^{-1}\| &= \|z([\varphi(A^q)]^{1/q} + z)^{-1}\| \\ &\leq \frac{\sin(\pi/q)}{\pi/q} \frac{(\pi/q + \gamma)}{\sin(\pi/q + \gamma)} \sup_{s>0} \|s(\varphi(A^q) + s)^{-1}\| \\ &\leq \frac{\sin(\pi/q)}{\pi/q} \frac{(\pi/q + \gamma)}{\sin(\pi/q + \gamma)} \sup_{s>0} \|s(A^q + s)^{-1}\| \\ &\leq \tilde{M}_{\alpha,q;\gamma}(A) \frac{\sin(\pi/q)}{\pi/q} \frac{\pi/q + \gamma}{\sin(\pi/q + \gamma)}, \end{aligned}$$

where $\tilde{M}_{\alpha,q;\gamma}(A)$ is defined by (5.6).

Once again, since the choice of $q \in (1, \pi/\alpha)$ and $\gamma \in (0, (1 - q^{-1})\pi)$ is arbitrary, it follows that (5.9) holds for $z \in \Sigma_{\pi-\alpha'}$ for any $\alpha' \in (\alpha, \pi)$. \square

6. MAIN RESULTS: HOLOMORPHICITY AND PRESERVATION OF ANGLE

We start with recalling some basics on holomorphic semigroups.

A C_0 -semigroup $(e^{-tA})_{t \geq 0}$ is said to be a holomorphic semigroup of angle θ if $e^{-\cdot A}$ extends holomorphically to a sector Σ_θ for some $\theta \in (0, \frac{\pi}{2}]$. In this case, we write $-A \in \mathcal{H}(\theta)$. If this extension is bounded in $\Sigma_{\theta'}$ for every $\theta' \in (0, \theta)$ then we say that $(e^{-tA})_{t \geq 0}$ is a sectorially bounded holomorphic semigroup of angle θ and write $-A \in \mathcal{BH}(\theta)$. Note that $(e^{-tA})_{t \geq 0}$ may admit a holomorphic extension to Σ_θ as above without being sectorially bounded (as already one-dimensional examples show). Recall that if $\omega \in [0, \pi/2)$ then $A \in \text{Sect}(\omega)$ if and only if $-A \in \mathcal{BH}(\pi/2 - \omega)$, see e.g. [11, Theorem 4.6].

Berg, Boyadzhiev and de Laubenfels proved in [4, Propositions 7.1 and 7.4] that if $-A \in \mathcal{BH}(\theta)$ and $\theta \in (\pi/4, \pi/2]$, then for any $\psi \in \mathcal{BF}$ the operator $-\psi(A)$ generates a sectorially bounded holomorphic C_0 -semigroup, and if $-A \in \mathcal{BH}(\pi/2)$, then $-\psi(A) \in \mathcal{BH}(\pi/2)$ too. They also asked in [4] whether the statement holds for θ from the whole of the interval $(0, \pi/2]$. In Theorem 6.4 below, we remove the restriction on θ and prove the result in full generality thus solving a problem posed in [4]. Moreover, we show that ψ the holomorphy angle of $(e^{-tA})_{t \geq 0}$ is invariant. As byproduct, in Corollary 6.6, we also answer the question by Kishimoto-Robinson from [28] mentioned in Introduction. To this aim, we will first need to prove several results on functional calculi allowing one to apply the estimate (3.35) proved in Theorem 3.11.

Let $-A \in \mathcal{BH}(\theta)$ for some $\theta \in (0, \pi/2]$ so that for every $\omega \in (0, \pi/2 + \theta)$,

$$(6.1) \quad \|(z + A)^{-1}\| \leq \frac{M(A, \omega)}{|z|}, \quad z \in \Sigma_\omega.$$

The following assumption will be crucial:

For the whole of this section, let $\omega \in (\pi/2, \pi/2 + \theta)$ be fixed. Let also ψ be a Bernstein function, φ be the complete Bernstein function associated to ψ , and the function r be given by (3.34).

Define $r(A, \cdot) : \Sigma_\omega \rightarrow \mathcal{L}(X)$ and $F(A, \cdot) : \Sigma_\omega \rightarrow \mathcal{L}(X)$ by

$$(6.2) \quad r(A; z) := \frac{1}{2\pi i} \int_{\partial \Sigma_\beta} r(\lambda; z)(\lambda - A)^{-1} d\lambda,$$

and

$$(6.3) \quad F(A; z) := \frac{1}{2\pi i} \int_{\partial \Sigma_\beta} \frac{\lambda r(\lambda; z)}{(\lambda + 1)^2} (\lambda - A)^{-1} d\lambda,$$

where $\beta \in (\pi/2 - \theta, \pi - \omega)$ is arbitrary and Σ_β is oriented counterclockwise. In view of (3.35) and Cauchy's theorem, the functions r and F are well-defined.

We start with providing sectoriality estimates for r in appropriate sectors an expressing F via r .

Proposition 6.1. *Let $-A \in \mathcal{BH}(\theta)$ for some $\theta \in (0, \pi/2]$ so that (6.1) holds. Then for every $\omega \in (\pi/2, \pi/2 + \theta)$, $r(A, \cdot)$ is holomorphic in Σ_ω and for every $\beta \in (\pi/2 - \theta, \pi - \omega)$,*

$$(6.4) \quad \|r(A; z)\| \leq \frac{4M(A, \pi - \beta)}{\pi \cos^2 \beta \cos^2((\gamma + \beta)/2) |z|}, \quad z \in \Sigma_\omega.$$

In particular, if $\delta := \pi/2 + \theta - \omega$, then

$$(6.5) \quad \|r(A; z)\| \leq \frac{4M(A, \pi/2 + \theta - \delta/2)}{\pi \sin^2(\theta/2) \sin^2(\delta/4) |z|}, \quad z \in \Sigma_\omega.$$

Proof. The estimate (6.4) follows from (6.2), (3.35) and (6.1). Setting

$$\delta = \pi/2 + \theta - \omega, \quad \beta = \pi/2 - \theta + \delta/2,$$

in (6.4), we obtain (6.5).

The holomorphicity of $r(A, \cdot)$ in Σ_γ is a direct consequence of Fubini's and Morera's theorems. \square

Lemma 6.2. *Let $r(A; z)$ and $F(A; z)$ be defined by (6.2) and (6.3), respectively. Then*

$$(6.6) \quad F(A; z) = A(A + 1)^{-2} r(A; z), \quad z \in \Sigma_\omega.$$

Proof. Note that

$$\begin{aligned} \frac{\lambda}{(\lambda + 1)^2} - A(A + 1)^{-2} &= [\lambda(A + 1)^2 - (\lambda + 1)^2 A] \frac{(A + 1)^{-2}}{(\lambda + 1)^2} \\ &= (\lambda A - 1)(A - \lambda) \frac{(A + 1)^{-2}}{(\lambda + 1)^2}, \quad \lambda \in \mathbb{C} \setminus (-\infty, 0). \end{aligned}$$

Therefore, by (3.39), for every $x \in X$ one has

$$\begin{aligned}
& F(A; z)x - A(A+1)^{-2}r(A; z)x \\
&= \frac{1}{2\pi i} \int_{\partial\Sigma_\beta} r(\lambda; z) \left[\frac{\lambda}{(1+\lambda)^2} - A(A+1)^{-2} \right] (\lambda - A)^{-1}x d\lambda \\
&= -\frac{1}{2\pi i} \int_{\partial\Sigma_\beta} \frac{r(\lambda; z)}{(\lambda+1)^2} (\lambda A - 1)(A+1)^{-2}x d\lambda \\
&= \frac{1}{2\pi i} \int_{\partial\Sigma_\beta} \frac{r(\lambda; z)}{(\lambda+1)^2} (A+1)^{-2}x d\lambda \\
&\quad - \frac{1}{2\pi i} \int_{\partial\Sigma_\beta} \frac{\lambda r(\lambda; z)}{(\lambda+1)^2} A(A+1)^{-2}x d\lambda \\
&= 0.
\end{aligned}$$

□

The following statement relating the resolvents of $\psi(A)$ and $\varphi(A)$ will be basic for proving the main result of this paper, Theorem 6.4. It shows that the resolvents do not differ much as far their behavior at infinity is concerned.

Proposition 6.3. *Let $-A \in \mathcal{BH}(\theta)$ for some $\theta \in (0, \pi/2]$. Then*

$$(6.7) \quad (z + \psi(A))^{-1} = (z + \varphi(A))^{-1} + r(A; z), \quad z \in \Sigma_\omega.$$

Proof. Suppose first that A has dense range. Then the operators $(s + \psi)^{-1}(A)$ and $(s + \varphi)^{-1}(A)$ are well-defined for $s > 0$ via the (extended) holomorphic functional calculus with the regulariser τ given by (4.4). On the other hand, since $-\psi(A)$ and $-\varphi(A)$ generate bounded C_0 -semigroups, we have that

$$(s + \psi(A))^{-1} \in \mathcal{L}(X) \quad \text{and} \quad (s + \varphi(A))^{-1} \in \mathcal{L}(X), \quad s > 0.$$

Moreover, by [20, Theorem 1.3.2, f)], if $s > 0$, then

$$(s + \psi(A))^{-1} = (s + \psi)^{-1}(A) \quad \text{and} \quad (s + \varphi(A))^{-1} = (s + \varphi)^{-1}(A).$$

Hence, using sum rule for the (extended) holomorphic functional calculus,

$$(s + \psi(A))^{-1} - (s + \varphi(A))^{-1} = [(s + \psi)^{-1} - (s + \varphi)^{-1}](A).$$

Furthermore, using the holomorphic functional calculus once again,

$$\begin{aligned}
(s + \psi(A))^{-1} - (s + \varphi(A))^{-1} &= [\tau(A)]^{-1} [((s + \psi)^{-1} - (s + \varphi)^{-1})\tau](A) \\
&= [A(A+1)^{-2}]^{-1} [r(s; \cdot)\tau](A) \\
&= [A(A+1)^{-2}]^{-1} F(A; s).
\end{aligned}$$

From this and (6.6) it follows that

$$(6.8) \quad (s + \psi(A))^{-1} = (s + \varphi(A))^{-1} + r(A; s), \quad s > 0,$$

that is (6.7) holds for $z > 0$.

To obtain (6.8) in case when the range of A may not be dense, we consider the approximation of A by the operators A_ϵ with dense range given by

$$A_\epsilon := A + \epsilon \in \mathcal{BH}(\theta), \quad \epsilon > 0.$$

By (6.8) we have

$$(6.9) \quad (s + \psi(A_\epsilon))^{-1} - (s + \varphi(A_\epsilon))^{-1} = r(A_\epsilon; s), \quad s > 0, \quad \epsilon > 0.$$

Using Proposition 4.13 now, we can apply (4.19) to the Bernstein functions ψ and φ . It follows that

$$\lim_{\epsilon \rightarrow 0} [(s + \psi(A_\epsilon))^{-1} - (s + \varphi(A_\epsilon))^{-1}] = (s + \psi(A))^{-1} - (s + \varphi(A))^{-1},$$

in the uniform operator topology. On the other hand, by (3.5),

$$|\lambda + \epsilon| \geq \cos(\beta/2) (|\lambda| + \epsilon), \quad \lambda \in \partial\Sigma_\beta, \quad \epsilon > 0.$$

Therefore, if $\lambda \in \partial\Sigma_\beta$ then

$$\|(A - \lambda)^{-1} - (A - \lambda - \epsilon)^{-1}\| \leq \epsilon \frac{M^2(\pi - \beta, A)}{|\lambda(\lambda + \epsilon)|} \leq \frac{\epsilon M^2(\pi - \beta, A)}{\cos(\beta/2) |\lambda| (|\lambda| + \epsilon)}.$$

So, by (6.2), (3.35) and the bounded convergence theorem, we obtain that

$$\begin{aligned} \|r(A; s) - r(A_\epsilon; s)\| &\leq \frac{1}{2\pi} \int_{\partial\Sigma_\beta} r(\lambda; z) \|(A - \lambda)^{-1} - (A - \lambda - \epsilon)^{-1}\| |d\lambda| \\ &\leq \frac{\epsilon M^2(\pi - \beta, A)}{2\pi} \int_{\partial\Sigma_\beta} \frac{|r(\lambda; s)|}{|\lambda| (|\lambda| + \epsilon)} |d\lambda| \\ &\rightarrow 0, \quad \epsilon \rightarrow 0. \end{aligned}$$

Letting $\epsilon \rightarrow 0$ in (6.9), (6.8) follows.

Thus, $(\cdot + \psi(A))^{-1}$ satisfies (6.8) and extends holomorphically to Σ_ω as both $r(\cdot, A)$ and $(\cdot + \varphi(A))^{-1}$ have the latter property by Proposition 6.1 and Theorem 5.3, respectively. Then, [2, Appendix B, Proposition B5] implies that $\Sigma_\omega \subset \rho(-\psi(A))$ and the extension is given by $(\cdot + \psi(A))^{-1}$. This yields (6.7) for all $z \in \Sigma_\omega$. \square

Now we are ready to prove the main results of this paper. under Bernstein functions. The first statement of them shows that Bernstein functions leave the class of generators of sectorially bounded holomorphic semigroups on a Banach space invariant and, moreover, preserve the holomorphy sectors.

Theorem 6.4. *Let $-A \in \mathcal{BH}(\theta)$ for some $\theta \in (0, \pi/2]$. Then for every $\psi \in \mathcal{BF}$ one has $-\psi(A) \in \mathcal{BH}(\theta)$. Moreover, if*

$$(6.10) \quad \alpha = \pi/2 - \theta, \quad 2 < q < \pi/\alpha, \quad \text{and} \quad \pi/2 < \gamma < (1 - q^{-1})\pi,$$

then

$$(6.11) \quad \|z(z + \psi(A))^{-1}\| \leq \tilde{C}_{q,\gamma}(\theta), \quad z \in \Sigma_\gamma,$$

where

$$(6.12) \quad \tilde{C}_{q,\gamma}(\theta) = \frac{\tilde{M}_{\alpha,q}(A) \sin(\pi/q)}{\pi/q} \frac{\pi/q + \gamma}{\sin(\pi/q + \gamma)} + \frac{4M(A, \pi/2 + \theta - \delta/2)}{\pi \sin^2(\theta/2) \sin^2(\delta/4)},$$

and $\delta = \pi/2 + \theta - \gamma$.

Proof. Let $\varphi \in \mathcal{CBF}$ be the function associated with $\psi \in \mathcal{BF}$. By Theorem 5.3, if $-A \in \mathcal{BH}(\theta)$ then $-\varphi(A) \in \mathcal{BH}(\theta)$. Moreover, if q and γ satisfying (6.10) are fixed, then by (6.5),

$$(6.13) \quad \|zr(A; z)\| \leq \frac{4M(A, \pi/2 + \theta - \delta/2)}{\pi \sin^2(\theta/2) \sin^2(\delta/4)}, \quad z \in \Sigma_\gamma,$$

where $\delta = \pi/2 + \theta - \gamma$. Hence from Proposition 6.3, Theorem 5.3 and (6.13) it follows that (6.11) holds with the constant $\tilde{C}_{q,\gamma}(\theta)$ defined by (6.12). Since the choice of q and γ satisfying (6.10) is arbitrary, and γ is arbitrarily close to $\pi - \alpha = \pi/2 + \theta$ if q is sufficiently close to π/α , we conclude that $-\psi(A) \in \mathcal{BH}(\theta)$. \square

Theorem 6.4 has a version saying that Bernstein functions preserve the class of bounded (but not necessarily sectorially) holomorphic C_0 -semigroups. This version is an immediate consequence of Theorem 6.4 and the following lemma.

Lemma 6.5. *Let $-A$ be the generator of a bounded C_0 -semigroup on X and let ψ be a Bernstein function. Suppose there exists $d \geq 0$ such that $-\psi(A + d) \in \mathcal{H}(\theta)$ for some $\theta \in (0, \pi/2]$. Then $-\psi(A) \in \mathcal{H}(\theta)$.*

Proof. By Proposition 4.10, (i), we have

$$(6.14) \quad \psi(A + d) = \psi(A) + [\psi(\cdot + d) - \psi(\cdot)](A) = \psi(A) + B_d.$$

By the product rule for the (extended) Hille-Phillips calculus we have

$$(\psi(A + d) + s)^{-1}(B_d + s)^{-1} = (B_d + s)^{-1}(\psi(A + d) + s)^{-1}$$

for sufficiently large $s > 0$. Then, by [37, Section A-I.3.8, p. 24] (see also [1, Theorem 1]) it follows that the C_0 -semigroups $(e^{-t\psi(A+d)})_{t \geq 0}$ and $(e^{-tB_d})_{t \geq 0}$ commute. Then, taking into account that $\text{dom}(\psi(A)) = \text{dom}(\psi(A + d))$ by Proposition 4.10, (ii) and using [11, Subsection II.2.7], we conclude that

$$(6.15) \quad e^{-t\psi(A)} = e^{-t\psi(A+d)} e^{tB_d}, \quad t > 0.$$

Since $(e^{tB_d})_{t \geq 0}$ extends to an entire function, the statement of lemma follows. \square

Corollary 6.6. *Let $-A$ be the generator of a bounded C_0 -semigroup on X such that $-A \in \mathcal{H}(\theta)$ for some $\theta \in (0, \pi/2]$. Then for every $\psi \in \mathcal{BF}$ one has $-\psi(A) \in \mathcal{H}(\theta)$.*

Proof. Observe that if $(e^{-tA})_{t \geq 0}$ is a bounded C_0 -semigroup admitting a holomorphic extension to Σ_θ , $\theta \in (0, \pi/2]$, then by e.g. [2, Proposition 3.7.2 b)] we infer that for fixed $\theta' \in (0, \theta)$ and big enough $d > 0$ the operator $-(d + A)$ generates a C_0 -semigroup $(e^{-t(d+A)})_{t \geq 0}$ which is holomorphic and sectorially bounded in $\Sigma_{\theta'}$. Then, by Theorem 6.4 the C_0 -semigroup $(e^{-t\psi(d+A)})_{t \geq 0}$ is also holomorphic and sectorially bounded in

$\Sigma_{\theta'}$. By Lemma 6.5, $-\psi(A)$ generates a bounded C_0 -semigroup which extends holomorphically to $\Sigma_{\theta'}$. Since the choice of $\theta' \in (0, \theta)$ is arbitrary, the corollary follows. \square

Remark 6.7. It was claimed in [34] that if $-A$ is the generator of a bounded C_0 -semigroup on X then $-A \in \cup_{\theta \in (0, \pi/2]} \mathcal{H}(\theta)$ implies the same property for $-\psi(A)$. Unfortunately, the proof of this fact in [34] seems to contain a mistake. Specifically, in the notation of [34], the proof relies on the boundedness of the operator $\psi(A)g_t(A)$ which not proved in [34]. (In fact, it is easy to show that the boundedness of $\psi(A)g_t(A)$ is equivalent to the holomorphicity of $(e^{-t\psi(A)})_{t \geq 0}$). Nonetheless, the holomorphicity of $(e^{-t\psi(A)})_{t \geq 0}$ was proved in [33] for uniformly convex X by means of Kato-Pazy's criterion, see [27] and [39, Corollaries 2.5.7 and 2.5.8].

Let $-A$ be the generator of a bounded C_0 -semigroup *on* with dense range (and then trivial kernel by the mean ergodic theorem). Consider so-called Stieltjes functions $f : (0, \infty) \rightarrow (0, \infty)$ which can be defined by the property that $1/f \in \mathcal{CBF}$. Recall also that for $f \in \mathcal{CBF}$ one has that $1/f$ is Stieltjes so that the class of Stieltjes functions is, in a sense, a reciprocal dual of the class of complete Bernstein functions. Note that for Stieltjes f the operator $-f(A)$, does not, in general, generate a C_0 -semigroup. For example, if $f(z) = 1/z$ then $f(A) = A^{-1}$, and the corresponding counterexample can be found in [15]. On the other hand, for generators of sectorially bounded holomorphic C_0 -semigroups the situation is different and we have the following statement, which follows from Theorem 6.4, (4.25) and the fact that inverses of generators of bounded holomorphic C_0 -semigroups of angle θ generate semigroups of the same kind. (For the latter statement see e.g. [10].)

Recall that $f \in \mathcal{P}$ if there exists a nonzero $\psi \in \mathcal{BF}$ such that $f = 1/\psi$. Note also that by the discussion preceding Proposition 4.11, for $f \in \mathcal{P}$ the operator $f(A)$ is well-defined in the (extended) holomorphic functional calculus for $A \in \text{Sect}(\theta)$, $\theta \in (0, \pi/2]$, having dense range.

Theorem 6.8. *Suppose that $-A \in \mathcal{BH}(\theta)$ for some $\theta \in (0, \pi/2]$ and $\text{ran}(A)$ is dense. Then for every $f \in \mathcal{P}$ one has $-f(A) \in \mathcal{BH}(\theta)$.*

Now we can extend the classes of admissible ψ and f in Theorems 6.4 and 6.8.

Corollary 6.9. *Suppose that $-A \in \mathcal{BH}(\theta)$ for some $\theta \in (0, \pi/2]$ and $\text{ran}(A)$ is dense. If $h = \psi + f$, where $\psi \in \mathcal{BF}$ and $f \in \mathcal{P}$, then $-h(A) \in \mathcal{BH}(\theta)$.*

Proof. By Theorems 6.4 and 6.8, we have $-\psi(A) \in \mathcal{BH}(\theta)$, $-f(A) \in \mathcal{BH}(\theta)$. By the product rule for the (extended) holomorphic functional calculus it follows that for every $s > 0$:

$$\begin{aligned} [(s + \psi(\cdot))^{-1}(s + f(\cdot))^{-1}](A) &= (s + \psi(A))^{-1}(s + f(A))^{-1} \\ &= (s + f(A))^{-1}(s + \psi(A))^{-1}. \end{aligned}$$

Hence, as in the proof of Lemma 6.5, the semigroups $(e^{-t\psi(A)})_{t \geq 0}$ and $(e^{-tf(A)})_{t \geq 0}$ commute. Then, by [11, Subsection II.2.7], $-\psi(A) - f(A)$ generates a C_0 -semigroup $(e^{-t\psi(A)}e^{-tf(A)})_{t \geq 0}$, and therefore $-\psi(A) - f(A) \in \mathcal{BH}(\theta)$. From this, by Proposition 4.11, it follows $-h(A) \in \mathcal{BH}(\theta)$. \square

Note that in the particular case when $\varphi \in \mathcal{CBF}$ and f is a Stieltjes function (i.e. $1/f \in \mathcal{CBF}$), Corollary 6.8 was proved in [4, Theorem 6.4].

7. IMPROVING PROPERTIES OF BERNSTEIN FUNCTIONS: CARASSO-KATO FUNCTIONS

Let us first recall some notions and results from [6]. To this aim and for formulating our results in this section the next definition will be helpful.

Definition 7.1. A Bernstein function ψ is said to be Carasso-Kato if for every Banach space X , and every bounded C_0 -semigroup $(e^{-tA})_{t \geq 0}$ on X , the C_0 -semigroup $(e^{-t\psi(A)})_{t \geq 0}$ is holomorphic.

Following [6], denote the set of vaguely continuous convolution semigroups of subprobability measures on $[0, \infty)$ by \mathcal{T} . Let \mathcal{I} stand for the set of $(\mu_t)_{t \geq 0} \in \mathcal{T}$ such that a Bernstein function ψ given by $(\mu_t)_{t \geq 0}$ via Bochner's formula (3.42) is Carraso-Kato. Let us finally denote by $\mathcal{T}_1 \subset \mathcal{T}$ the set of functions $[0, \infty) \ni t \mapsto \mu_t$ such that μ_t is continuously differentiable in $M_b([0, \infty))$ for $t > 0$, with

$$\|\mu_t'\|_{M_b} = O(t^{-1}) \quad \text{as } t \rightarrow 0+.$$

Recall that by [6, Theorem 4],

$$(7.1) \quad \mathcal{I} = \mathcal{T}_1.$$

Moreover, $(\mu_t)_{t \geq 0} \in \mathcal{I}$ implies that $\psi \in \mathcal{BF}$ defined by (3.42) satisfies

$$(7.2) \quad \psi(\mathbb{C}_+) \subset \overline{\Sigma}_\gamma - \beta := \{\lambda \in \mathbb{C} : \lambda + \beta \in \overline{\Sigma}_\gamma\}$$

for some $\gamma \in (0, \pi/2)$ and $\beta \geq 0$. Hence, as it was shown in [6], there exists $K > 0$ such that

$$|\psi(z)| \leq K|z|^{2\gamma/\pi}, \quad |z| \geq 1, \quad z \in \mathbb{C}_+.$$

While [6] describes Carraso-Kato functions ψ in terms of the families of measures $(\mu_t)_{t \geq 0}$ corresponding to ψ via (3.42), the results of [6] are not so easy to apply since one is usually given ψ rather than the corresponding family $(\mu_t)_{t \geq 0}$. The aim of this section is to single out substantial classes of Carraso-Kato functions ψ in terms of geometric properties of ψ themselves.

Note first that Corollary 6.6 yields immediately the following assertion.

Corollary 7.2. *Let $\psi \in \mathcal{BF}$ and let φ be a Carasso-Kato function. Then $\psi \circ \varphi$ is also Carasso-Kato.*

Remark 7.3. Let $\psi, \varphi \in \mathcal{BF}$, so that

$$e^{-t\psi(z)} = \int_0^\infty e^{-zs} \mu_t(ds), \quad e^{-t\varphi(z)} = \int_0^\infty e^{-zs} \nu_t(ds), \quad z \geq 0, \quad t \geq 0,$$

for some $(\mu_t)_{t \geq 0}, (\nu_t)_{t \geq 0} \in \mathcal{T}$. Then, according to [45, Theorem 5.19],

$$e^{-t(\psi \circ \varphi)(z)} = \int_0^\infty e^{-z\tau} \eta_t(d\tau),$$

where $(\eta_t)_{t \geq 0} \in \mathcal{T}$ is given by a convolution formula

$$(7.3) \quad \eta_t(d\tau) = \int_0^\infty \nu_s(d\tau) \mu_t(ds).$$

Thus, in the situation of Corollary 7.2, the property (7.1) implies that $(\eta_t)_{t \geq 0}$ belongs to \mathcal{T}_1 .

Example 7.4. a) It was proved in [6] that the function

$$\varphi(z) = \log(z+1), \quad z \geq 0,$$

is Carasso-Kato. If $-A$ is the generator of a bounded C_0 -semigroup, then as $\varphi \in \mathcal{CBF}$, Hirsch's calculus yields

$$\log(1+A)x = \int_1^\infty s^{-1}(s+A)^{-1}Ax \, ds, \quad x \in \text{dom}(A).$$

By the improving property of φ , $-\log(1+A)$ is the generator of a bounded holomorphic C_0 -semigroup $(e^{-t\log(1+A)})_{t \geq 0}$ on X given by

$$e^{-t\log(1+A)} = (1+A)^{-t} = \int_0^\infty e^{-sA} e^{-s} \frac{s^{t-1}}{\Gamma(t)} \, ds, \quad t \geq 0,$$

so that $\varphi \in \mathcal{BF}$ corresponds to the semigroup

$$\nu_t(ds) = \frac{s^{t-1}e^{-s}}{\Gamma(t)} \, ds, \quad t > 0,$$

via (3.13). This was proved in [6] with a quite complicated argument. From Corollary 7.2 we infer that for every $(\mu_t)_{t \geq 0} \in \mathcal{T}$,

$$\eta_t(d\tau) = \left(\int_0^\infty \frac{\tau^{s-1}e^{-s}}{\Gamma(s)} \mu_t(ds) \right) d\tau \in \mathcal{T}_1.$$

b) Consider a complete Bernstein function $\varphi(z) = \sqrt{z}$. Observe that

$$e^{-s\varphi(z)} = e^{-sz^{1/2}} = \frac{s}{2\sqrt{\pi}} \int_0^\infty e^{-zr} \frac{e^{-s^2/4r}}{r^{3/2}} \, dr,$$

and it easy to check that φ is Carraso-Kato, [6]. By Corollary 7.2, for each $(\mu_t)_{t \geq 0} \in \mathcal{T}$,

$$\eta_t(dr) = \frac{1}{2\sqrt{\pi}} \left(\int_0^\infty s e^{-s^2/4r} \mu_t(ds) \right) \frac{dr}{r^{3/2}} \in \mathcal{T}_1.$$

We proceed with several new conditions for a function to be Carraso-Kato. Roughly, they say that the function is Carasso-Kato if it shrinks a large angular sector to a smaller one.

The first statement provides a geometric condition for a stronger version of the Carraso-Kato property.

Theorem 7.5. *Let ψ be a Bernstein function. Suppose there exist $\theta_1 \in (0, \pi)$ and $\theta_2 \in (\pi/2, \pi)$ such that ψ admits a continuous extension $\tilde{\psi}$ to $\overline{\Sigma}_{\theta_2}$ which is holomorphic in Σ_{θ_2} , and*

$$(7.4) \quad \tilde{\psi}(\overline{\Sigma}_{\theta_2}^+) \subset \overline{\Sigma}_{\theta_1}^+.$$

Then the following holds.

- (i) *One has $\tilde{\psi} \in \mathcal{B}(\Sigma_{\theta_2})$ so that if $A \in \text{Sect}(\gamma)$, $\gamma < \theta_2$, is injective then $\tilde{\psi}(A)$ is well-defined in the (extended) holomorphic functional calculus.*
- (ii) *One has*

$$\tilde{\psi}(A) \in \text{Sect}(\omega), \quad \omega = \gamma \cdot \frac{\theta_1}{\theta_2}.$$

In particular, if $0\gamma \in (0, \frac{\pi\theta_2}{2\theta_1})$ then

$$(7.5) \quad -\tilde{\psi}(A) \in \mathcal{BH}(\theta), \quad \theta = \frac{\pi}{2} \left(1 - \frac{2\gamma}{\pi} \cdot \frac{\theta_1}{\theta_2} \right).$$

Proof. Let

$$\alpha := \frac{\theta_2}{\pi} \in (1/2, 1), \quad \beta := \frac{\pi}{\theta_1} > 1.$$

Then, by (7.4), both functions

$$\tilde{\psi}_\alpha(\lambda) := \tilde{\psi}(\lambda^\alpha), \quad \varphi(\lambda) := [\tilde{\psi}_\alpha(\lambda)]^\beta, \quad \lambda \in \mathbb{C} \setminus (-\infty, 0),$$

map the upper half-plane H^+ into itself. Hence, using Theorem 3.5, (iii), we conclude that

$$(7.6) \quad \tilde{\psi}_\alpha \in \mathcal{CBF}, \quad \varphi \in \mathcal{CBF}.$$

In particular, from the first inclusion in (7.6) and Lemma 3.4, (ii), it follows that $\tilde{\psi} \in \mathcal{B}(\Sigma_{\theta_2})$. Therefore, as $A \in \text{Sect}(\gamma)$ with $\gamma < \theta_2$, the operator $\tilde{\psi}(A)$ is well-defined in the (extended) holomorphic functional calculus, and (i) is proved.

Since $\gamma/\alpha < \pi$, Proposition 4.5 implies that

$$(7.7) \quad A^{1/\alpha} \in \text{Sect}(\gamma/\alpha).$$

Moreover, if $\gamma_0 \in (\gamma, \pi)$ is fixed and γ' satisfies $\gamma < \gamma_0 < \gamma' < \pi\alpha$ then using Proposition 5.2 it follows that

$$M(A^{1/\alpha}, \gamma'/\alpha) = \sup_{z \in \Sigma_{\pi-\gamma'/\alpha}} \|z(z + A^{1/\alpha})^{-1}\| \leq CM(A, \gamma_0),$$

where the constant C does not depend on A . Recall that the Hirsch functional calculus and the (extended) holomorphic functional calculus are compatible by Proposition 4.12. Thus, since $\tilde{\psi}$ is complete Bernstein, by (4.11), Theorem 5.3 and (7.7),

$$(7.8) \quad \tilde{\psi}_\alpha(A^{1/\alpha}) = [\varphi(A^{1/\alpha})]^{1/\beta} \in \text{Sect}(\gamma/(\alpha\beta)),$$

where the operators are defined by the Hirsch functional calculus.

Furthermore, if $\omega_0 \in (\omega, \pi)$ is fixed and ω' satisfies $\omega < \omega_0 < \omega' < \pi$, then Theorem 5.3 implies that

$$\|z(z + \tilde{\psi}_\alpha(A^{1/\alpha}))^{-1}\| \leq CM(A^{1/\alpha}, \omega_0), \quad z \in \Sigma_{\pi-\omega'},$$

where once again C does not depend on A .

Next, as A is injective, we use Propositions 4.12 and 4.4 to infer from (7.8) that

$$\tilde{\psi}(A) = \tilde{\psi}_\alpha(A^{1/\alpha}) \in \text{Sect}(\omega), \quad \omega = \frac{\gamma}{\alpha\beta} = \gamma \cdot \frac{\theta_1}{\theta_2},$$

where $\tilde{\psi}(A)$ is defined by the (extended) holomorphic functional calculus. In particular, if $\gamma \in (0, \frac{\pi\theta_2}{2\theta_1})$, then

$$-\tilde{\psi}(A) \in \mathcal{BH}(\theta), \quad \theta = \frac{\pi}{2} - \omega = \frac{\pi}{2} \left(1 - \frac{2\gamma}{\pi} \cdot \frac{\theta_1}{\theta_2}\right).$$

□

Corollary 7.6. *Let ψ be a Bernstein function satisfying the conditions of Theorem 7.5 where, in addition, $\theta_1 < \theta_2$. Let $-A$ be the generator of a bounded C_0 -semigroup on X . Then*

$$(7.9) \quad -\psi(A) \in \mathcal{BH}(\theta), \quad \theta = \frac{\pi}{2} \left(1 - \frac{\theta_1}{\theta_2}\right),$$

where $\psi(A)$ is given by the (extended) Hille-Phillips functional calculus. Moreover, if $-A \in \mathcal{BH}(\theta_0)$, $\theta_0 \in (0, \pi/2]$, then

$$(7.10) \quad -\psi(A) \in \mathcal{BH}(\theta), \quad \theta = \theta_0 + \left(\frac{\pi}{2} - \theta_0\right) \left(1 - \frac{\theta_1}{\theta_2}\right).$$

Proof. If operator A is injective, then (7.9) follows from Theorem 7.5 with $\gamma = \pi/2$ and Proposition 4.13.

Assume now that A is not injective. Then consider the injective operators A_ϵ defined by

$$A_\epsilon := A + \epsilon, \quad \epsilon > 0.$$

Using once again Theorem 7.5 with $\gamma = \pi/2$ and Proposition 4.13 we obtain that

$$(7.11) \quad -\psi(A + \epsilon) \in \mathcal{BH}(\theta), \quad \theta = \frac{\pi}{2} \left(1 - \frac{\theta_1}{\theta_2}\right).$$

Moreover, since for any $\omega' \in (\pi/2, \pi)$,

$$\sup_{z \in \Sigma_{\pi-\omega'}} \|z(z + A + \epsilon)^{-1}\| \leq \sup_{z \in \Sigma_{\pi-\omega'}} \|z(z + A)^{-1}\| = M(A, \omega'), \quad \epsilon > 0,$$

the proof of Theorem 7.5 implies that if $\omega = \pi/2 - \theta$ and $\omega_0 \in (\omega, \pi)$ then

$$\sup_{\epsilon \in (0,1)} \sup_{z \in \Sigma_{\pi-\omega_0}} \|z(z + \psi(A_\epsilon))^{-1}\| < \infty.$$

From this, the resolvent convergence given by (4.22), and Vitali's theorem it follows that

$$\sup_{z \in \Sigma_{\pi-\omega_0}} \|z(z + \psi(A))^{-1}\| < \infty,$$

for all $\omega_0 \in (\omega, \pi)$. In other words, $-\psi(A) \in \mathcal{BH}(\theta)$.

Finally, if $-A \in \mathcal{BH}(\theta_0)$, then $A \in \text{Sect}(\pi/2 - \theta_0)$ and, similarly, using (7.5) with $\gamma = \pi/2 - \theta_0$, we obtain (7.10). \square

In a manner similar to the proof of Corollary 7.6, Theorem 7.5 yields also the following assertion providing a geometric condition for the Carasso-Kato property. The statement is, in fact, Theorem 1.3 mentioned in Introduction.

Corollary 7.7. *Let ψ be a Bernstein function. Suppose there exist $\theta \in (\pi/2, \pi)$ and $r > 0$ such that ψ admits a continuous extension $\overline{\Sigma}_\theta$ which is holomorphic in Σ_θ , and*

$$(7.12) \quad \psi(\lambda) \in \overline{\Sigma}_{\pi/2}^+ \quad \text{for } \lambda \in \overline{\Sigma}_\theta^+, \quad |\lambda| \geq r.$$

Then ψ is Carasso-Kato function. Moreover, for any generator A of a bounded C_0 -semigroup on X , one has $-\psi(A) \in \mathcal{H}(\frac{\pi}{2}(1 - \frac{\pi}{2\theta}))$.

Proof. From (7.12) it follows that there exists $d > 0$ such that function ψ_d given by

$$\psi_d(\lambda) := \psi(\lambda + d), \quad \lambda \in \overline{\Sigma}_\theta,$$

is holomorphic in Σ_θ and continuous in $\overline{\Sigma}_\theta$, and it satisfies

$$(7.13) \quad \psi_d(\lambda) \in \overline{\Sigma}_{\pi/2}^{(+)} \quad \text{for } \lambda \in \overline{\Sigma}_\theta^{(+)}.$$

Therefore, by Corollary 7.6 with $\theta_2 = \theta$ and $\theta_1 = \pi/2$, if $-A$ is the generator of a bounded C_0 -semigroup then $-\psi_d(A) = -\psi(A + d) \in \mathcal{BH}(\theta_0)$, where

$$\theta_0 = \frac{\pi}{2} \left(1 - \frac{\pi}{2\theta}\right) \in \left(0, \frac{\pi}{2}\right).$$

Then, by Lemma 6.5, we conclude that $-\psi(A)$ generates a bounded C_0 -semigroup possessing holomorphic extension to Σ_{θ_0} . \square

Let us recall now the next result by Fujita [13, p. 337 and Lemma 2].

Theorem 7.8. *Let $\alpha \in (0, 1)$, $\theta_\alpha = \pi/(1 + \alpha)$ and $\Theta \in (\theta_\alpha, \pi)$. Let $\psi \in \mathcal{BF}$ satisfy the following conditions:*

(A1) *ψ admits a continuous extension to $\overline{\Sigma}_\Theta$ which is holomorphic in Σ_Θ .*

(A2) *one has*

$$(7.14) \quad \lim_{r \rightarrow \infty} \frac{\psi(re^{i\theta})}{\psi(r)} = e^{i\alpha\theta} \quad \text{uniformly in } |\theta| \leq \theta_0,$$

and ψ is regularly varying of order α ,

(A3) $\psi(re^{\pm i\theta_\alpha})/r$ are integrable in a right neighborhood of 0.

Then ψ is a Carasso-Kato function.

Note that Theorem 7.8 follows from Corollary 7.7. Moreover, Corollary 7.7 shows that one can omit the condition (A3) and the assumption that ψ is a regularly varying. Indeed from (A1), (A2) and the properties

$$\psi(r) > 0, \quad r > 0, \quad \alpha\Theta < \alpha\theta_\alpha = \frac{\pi\alpha}{1+\alpha} < \pi/2,$$

it follows that for a large $r > 0$,

$$\psi(\lambda) \in \overline{\Sigma}_{\alpha\Theta}^{(+)} \subset \overline{\Sigma}_{\pi/2}^+ \quad \text{for } \lambda \in \overline{\Sigma}_\Theta^+(r),$$

hence (7.12) holds.

Now we turn our attention to Carraso-Kato functions ψ which are, in addition, *complete* Bernstein functions. As in the situation of Theorem 7.5, we first require ψ to map generators of bounded semigroups into the generators of *sectorially bounded* holomorphic semigroups. Such ψ can, in fact, be *characterized* in an elegant way as the following statement shows. (It corresponds to Theorem 1.2 from Introduction.)

Theorem 7.9. *Let ψ be a complete Bernstein function and let $\gamma \in (0, \pi/2)$ be fixed. The the following assertions are equivalent.*

(i) *One has*

$$\psi(\overline{\mathbb{C}}_+) \subset \overline{\Sigma}_\gamma.$$

(ii) *For each (complex) Banach space X and each generator $-A$ of a bounded C_0 -semigroup on X , the operator $-\psi(A)$ belongs to $\mathcal{BH}(\pi/2 - \gamma)$.*

Proof. The implication (ii) \Rightarrow (i) follows from [6, Theorem 4] and its proof. So, it suffices to prove that (i) implies (ii).

Assume that (ii) is true. Then, by Corollary 7.6 and Proposition 3.6, we obtain that

$$(7.15) \quad -\psi(A) \in \mathcal{BH}(\theta), \quad \theta = \frac{\pi}{2} \left(1 - \frac{\tilde{\theta}_0}{\theta_0} \right),$$

where $\theta_0 \in (\pi/2, \pi)$ is such that

$$|\cos \theta_0| = \frac{\cot \gamma}{\cot \gamma + 1},$$

and $\tilde{\theta}_0 = \tilde{\theta}_0(\theta_0) \in (0, \pi/2)$, is defined by the equation

$$\cot \tilde{\theta}_0 = C(\theta_0), \quad C(\theta) := \frac{\cot \gamma + 1}{\sin \theta} \left(\frac{\cot \gamma}{\cot \gamma + 1} - |\cos \theta| \right).$$

Note that

$$\lim_{\theta_0 \rightarrow \pi/2} C(\theta_0) = \cot \gamma$$

and therefore

$$\lim_{\theta_0 \rightarrow \pi/2} \tilde{\theta}_0(\theta_0) = \gamma.$$

Thus considering θ_0 in (7.15) arbitrarily close to $\pi/2$, we obtain the assertion (ii). \square

Let us recall that (7.2) is necessary for $\psi \in \mathcal{BF}$ to be a Carasso-Kato function. The next statement show that if moreover $\psi \in \mathcal{CBF}$ then (7.2) is also sufficient thus providing a characterization of the Carasso-Kato property for complete Bernstein functions.

Corollary 7.10. *Let ψ be complete Bernstein function. Then ψ is Carasso-Kato if and only if there exist $\gamma \in (0, \pi/2)$ and $\beta \geq 0$ such that (7.2) holds. Moreover, if (7.2) holds and if $-A$ generates a bounded C_0 -semigroup on X , then $-\psi(A) \in \mathcal{H}(\pi/2 - \gamma)$.*

Proof. It is sufficient to show that (7.2) implies that ψ is Carasso-Kato. If (7.2) is satisfied, by applying Theorem 7.9 to $\psi_\beta \in \mathcal{CBF}$, $\psi_\beta(\lambda) := \psi(\lambda + \beta)$, we obtain that

$$-\psi_\beta(A) = -\psi(\beta + A) \in \mathcal{BH}(\theta), \quad \theta = \pi/2 - \gamma.$$

for any generator $-A$ of a bounded C_0 -semigroup on X . Then, using Lemma 6.5, we conclude that $-\psi(A)$ generates a bounded C_0 -semigroup having a holomorphic extension to Σ_θ . \square

Remark 7.11. Observe that $\psi(z) = \log(1 + z)$ is complete Bernstein and, moreover,

$$|\operatorname{Im}(\psi(z))| \leq \pi/2, \quad z \in \mathbb{C}_+.$$

Thus, for every $\gamma \in (0, \pi/2)$ there is $\beta > 0$ such that ψ satisfies (7.2). Then, by Corollary 7.10, we obtain that ψ is Carasso-Kato, and, moreover, for any generator $-A$ of a bounded C_0 -semigroup the semigroup generated by $-\log(A + I)$ admits a holomorphic extension to $\mathbb{C}_+ = \Sigma_{\pi/2}$. This fact complements Example 7.4, a).

8. APPENDIX

It is an open question whether for any $\alpha \in (0, 1)$,

$$(8.1) \quad \psi \in \mathcal{BF} \implies [\psi(\lambda^\alpha)]^{1/\alpha} \in \mathcal{BF}.$$

A positive answer to this question would allow one to apply the methods from [4] directly and to obtain the results from Section 6 in a comparatively simple way. Let us analyse the property (8.1) in some more details.

Apart from the situation described in (5.8), it is known that (8.1) is true if $\alpha = 1/n, n \in \mathbb{N}$ [4, Proposition 7.1] (see, also [42, Remark 12]). The following Lemma 8.3 generalizes [4, Proposition 7.1] and (5.8) in the case

when $\alpha \in (0, 1/2]$ and $\psi \in \mathcal{BF}$ and it extends these statements for any $\alpha \in (0, 1)$ if ψ is a so-called special Bernstein function. Recall that a non-zero Bernstein function ψ is said to be a special Bernstein function, if $z/\psi(z)$ is a Bernstein function.

Proposition 8.1. *Let ψ be a Bernstein function and let $\alpha \in (0, 1)$. For $\beta > 0$ define*

$$\tilde{\psi}_{\alpha, \beta}(z) := \left(\frac{\psi(z^\alpha)}{z^\alpha} \right)^\beta, \quad z > 0.$$

Then $\tilde{\psi}_{\alpha, \beta}$ is complete monotone for all $\alpha \in (0, 1/2]$ and $\beta > 0$. If ψ is a special Bernstein function then $\tilde{\psi}_{\alpha, \beta}$ is complete monotone for all $\alpha \in (0, 1)$ and $\beta > 0$.

Proof. If $\psi \in \mathcal{BF}$ and $\alpha \in (0, 1/2]$, then by [12, Proposition 3.6] one has

$$f_\alpha(z) := z^{1-\alpha}\psi(z^\alpha) \in \mathcal{CBF},$$

so that $z/f_\alpha(z) \in \mathcal{CBF}$ and, by [45, Theorem 3.6, (ii)], for any $\beta > 0$,

$$\left[\frac{\psi(z^\alpha)}{z^\alpha} \right]^\beta = \left[\frac{z^{1-\alpha}\psi(z^\alpha)}{z} \right]^\beta = \left[\frac{z}{f_\alpha(z)} \right]^{-\beta}$$

is complete monotone. Let now $\alpha \in (0, 1)$ and ψ be a special Bernstein function so that $\psi(z) = z/f(z)$, $f \in \mathcal{BF}$. Then $f(z^\alpha)$ is Bernstein, and by [45, Theorem 3.6, (ii)] the function $\tilde{\psi}_{\alpha, \beta}$ given by

$$\tilde{\psi}_{\alpha, \beta}(z) := [f(z^\alpha)]^{-\beta}, \quad z > 0,$$

is complete monotone for any $\beta > 0$. □

Remark 8.2. Let us note two partial cases of Proposition 8.1. Let $\beta = 1/\alpha - 1$ and

$$(8.2) \quad \tilde{\psi}_\alpha(z) := \left(\frac{\psi(z^\alpha)}{z^\alpha} \right)^{1/\alpha-1}, \quad \alpha \in (0, 1).$$

If $\psi \in \mathcal{BF}$ and $\alpha \in (0, 1/2]$ then $\tilde{\psi}_\alpha$ is completely monotone. If $\psi \in \mathcal{SBF}$ and $\alpha \in (0, 1]$ then $\tilde{\psi}_\alpha$ is completely monotone as well.

Lemma 8.3. *Let ψ be a Bernstein function and let*

$$\psi_\alpha(z) := [\psi(z^\alpha)]^{1/\alpha}, \quad \alpha \in (0, 1).$$

If $\alpha \in (0, 1/2]$, then $\psi_\alpha(z) \in \mathcal{BF}$. If $\alpha \in (0, 1)$ and in addition ψ is a special Bernstein function, then ψ_α is a special Bernstein function too.

Proof. We have

$$(8.3) \quad \psi'_\alpha(z) = \psi'(z^\alpha) \tilde{\psi}_\alpha(z), \quad z > 0,$$

where $\tilde{\psi}_\alpha$ is defined by (8.2). Then, since $\psi'(z^\alpha)$ is complete monotone (as composition of Bernstein and complete monotone functions, see [45, Theorem 3.7]), $\psi_\alpha \in \mathcal{BF}$ by Remark 8.2.

If $\psi \in \mathcal{SBF}$ and $\alpha \in (0, 1)$ then $\psi = z/f$ for some $f \in \mathcal{SBF}$. Hence $f_\alpha(z) := [f(z^{1/\alpha})]^\alpha \in \mathcal{BF}$, and then $\psi_\alpha(z) = \frac{z}{f_\alpha(z)} \in \mathcal{SBF}$. \square

Example 8.4. Note that (8.1) does not imply that ψ belongs to \mathcal{SBF} . For instance,

$$\psi(z) := 1 - \frac{1}{(1+z)^2} = \frac{z(2+z)}{(1+z)^2}, \quad z > 0,$$

is Bernstein, while $\psi \notin \mathcal{SBF}$. Indeed,

$$\frac{z}{\psi(z)} = \frac{(1+z)^2}{2+z} = z + \frac{1}{2+z} \notin \mathcal{BF}.$$

On the other hand, we have

$$\psi(z) = \frac{z}{f_1(z)f_2(z)},$$

where

$$f_1(z) = 1+z \in \mathcal{CBF}, \quad \text{and} \quad f_2(z) = \frac{1+z}{2+z} = 1 - \frac{1}{2+z} \in \mathcal{BF}.$$

Thus, for $\alpha \in (0, 1)$,

$$([\psi(z^\alpha)]^{1/\alpha})' = \frac{2}{(1+z^\alpha)^3} \cdot [f_1(z^\alpha)]^{-(1/\alpha-1)} \cdot [f_2(z^\alpha)]^{-(1/\alpha-1)}$$

is complete monotone, so that $[\psi(z^\alpha)]^{1/\alpha} \in \mathcal{BF}$ for any $\alpha \in (0, 1)$.

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FACULTY OF MATHEMATICS AND COMPUTER SCIENCE, NICOLAS COPERNICUS UNIVERSITY, UL. CHOPINA 12/18, 87-100 TORUŃ, POLAND
E-mail address: gomilko@mat.umk.pl

INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES, ŚNIADECKICH 8, 00-956 WARSZAWA, POLAND, AND FACULTY OF MATHEMATICS AND COMPUTER SCIENCE, NICOLAS COPERNICUS UNIVERSITY, UL. CHOPINA 12/18, 87-100 TORUŃ, POLAND
E-mail address: tomilov@mat.umk.pl